Short Wire Antennas: A Simplified Approach

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Part I: Scaling Arguments

0. Introduction:

How does a wire dipole antenna work? How do we find the resistance and the reactance? Why does the reactance vanish at an appropriate length or frequency?

In typical textbook treatments [1-3] these problems are approached indirectly: The resistance is estimated by calculating the fields from a known current distribution along the wire at infinite distance, finding the consequent radiated power by integrating the Poynting vector over an arbitrarily large sphere, and setting the result equal to the product I^2R to find the equivalent resistance (***radiation resistance***). The reactance is calculated by solving Pocklington's or Hallen's integral equation numerically (the finite-element solution to this problem is generally referred to as the Method of Moments), or by integrating the Poynting vector over the antenna surface and equating it to the delivered power (induced emf method). These approaches are of course perfectly correct, but perhaps more obscure than they need to be. In this article we shall try to illustrate a simpler and more direct way of understanding how short wire antennas, and by extension other small antennas, interact with traveling electromagnetic waves, in which we focus on the potentials that result directly from charges and currents.

We shall use only three pieces of basic physics:

- Time-delayed Coulomb's law: each element of charge contributes an electric potential ϕ inversely proportional to the distance from the point of measurement, where the charge is evaluated at an earlier time corresponding to the propagation delay:

$$\phi(r,t) = \frac{\mu_0 c^2}{4\pi} \iiint \frac{q\left(r', t - \frac{|r'-r|}{c}\right)}{|r'-r|} dv \qquad (1.1)$$

- Time-delayed Ampere's law: each element of current contributes a vector potential **A** inversely proportional to the distance, and time-delayed. The vector potential **A** is oriented in the same direction as the current.

$$\mathbf{A}(r,t)=\frac{\mu_0}{4\pi}\iiint\frac{J\left(r',t-\frac{|r'-r|}{c}\right)}{|r'-r|}dv \tag{1.2}$$

- The resulting electric field is the sum of the potential gradient and the time derivative of the vector potential:

$$\mathbf{E}=-\frac{\partial}{dt}\mathbf{A}-\nabla\phi=-i\omega\mathbf{A}-\nabla\phi \tag{1.3}$$

where the second step assumes harmonic time dependence. The voltage from one point to another is the line integral of the electric field. Note that no explicit use of the magnetic field is required; we can banish cross-products and curls from consideration.

1. A Wire in Space

Consider a wire suspended in space, with an impinging vector potential **A** and consequent electric field $-i\omega\mathbf{A}$ (figure 1). For reasonable wire thicknesses, we can assume that currents and charges are only present in a thin layer on the surface of the wire, and adjust themselves to ensure zero field well within the wire. How is this arranged?

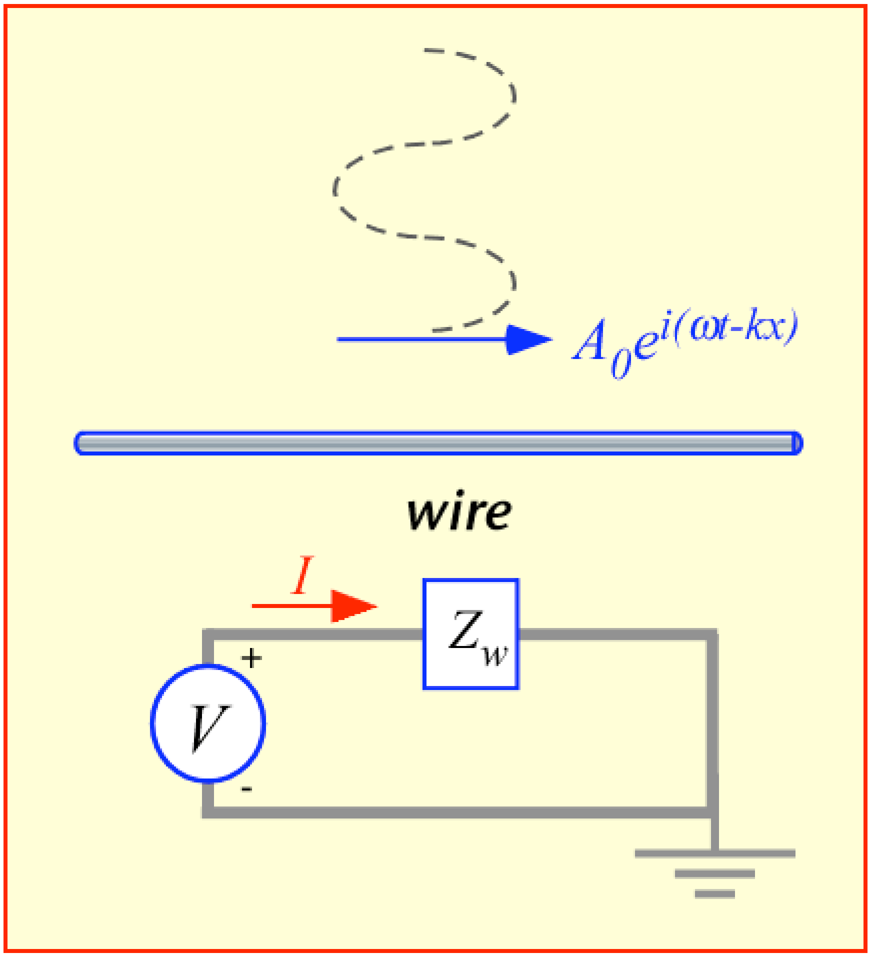

*Figure 1: wire and impinging potential; for simplicity, **A** is taken along the axis of the wire and the direction of propagation is perpendicular to the wire. Equivalent circuit depicts conventions for definition of voltage and current.*

To analyze the situation we begin with the basic source relationships: each infinitesimal volume of charge creates an electrostatic potential ϕ, decreasing inversely with distance and traveling at the speed of light. Similarly, each infinitesimal current element creates a vector potential **A**, oriented in the direction of the current, also decreasing inversely with distance and propagating at the speed of light.

The key to relating the current to the incident electric field is to decompose the ***scattered*** potential that arises due to current flow into three components, each of which has a distinct physical origin and differing dependence on the geometry of the wire:

- An instantaneous electric potential ϕ_{sc} results from the accumulation of charge, primarily near the ends of the wire, where charge must build up since current cannot flow past the ends. For short antennas we can ignore the time delay between the charge location *r'* and the axis of the wire *r*.

- An instantaneous magnetic component $A_{sc,in}$, in phase with the local current, whose value at each location is mainly determined by the current in that vicinity.
- A delayed magnetic component $A_{sc,d}$, dependent on the integral of the total current along the wire. For harmonic time dependence, this component lags the instantaneous component by 90 degrees – that is, it is along the negative imaginary axis (figure 2).

Corresponding to each scattered potential is an induced voltage. We shall somewhat arbitrarily choose to measure this voltage from left to right; the electrostatic voltage is thus the negative of the absolute potential, and the magnetic components are the line integral of the corresponding electric field along the wire. These three contributions are combined to obtain the net ***scattered voltage***; we then arrange the phase and amplitude of the current so that this scattered voltage cancels the incident voltage V_{inc}, ensuring that the interior of the wire is field-free as it must be.

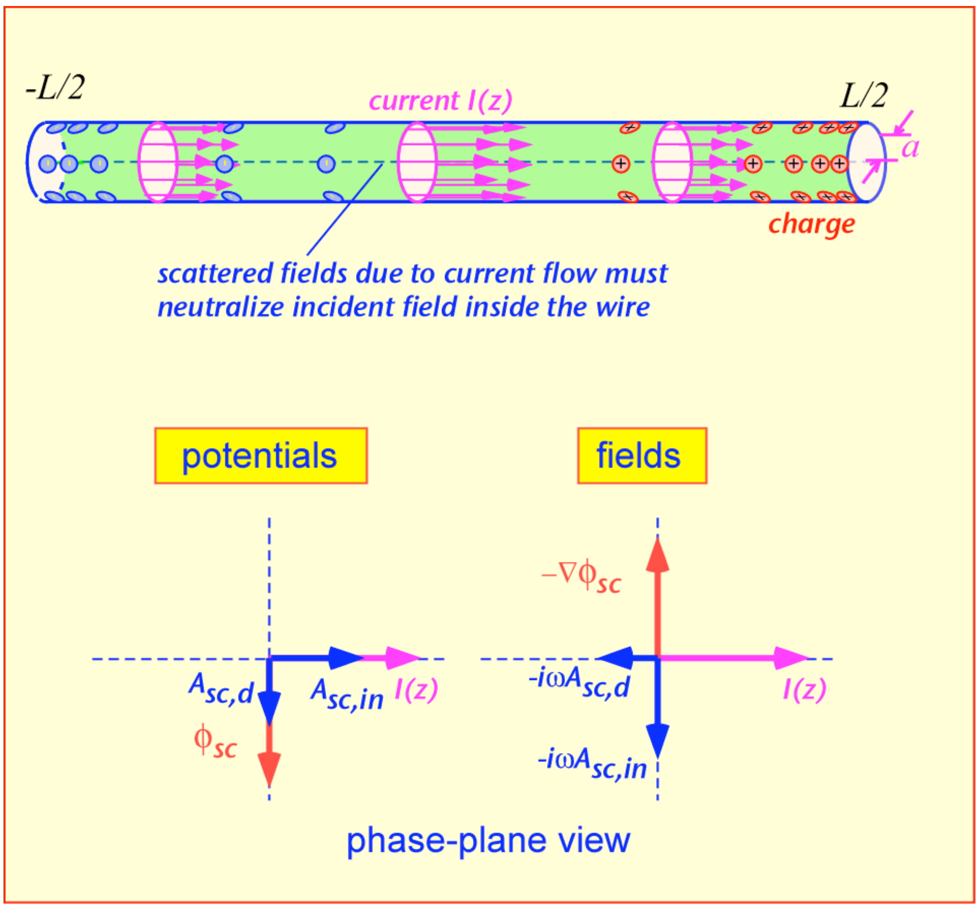

Figure 2: current and charge on the wire; resulting potentials and fields and their phase relationship with the current

Let's examine in a general way how each of these scattered components depends on the geometry of the wire. The total charge induced on each half of the wire is the integral of the current, from the simple relation that the current I is the charge Q divided by the time t:

$$I = \frac{Q}{t} \tag{1.4}$$

Thus by Coulomb's law, the potential is roughly (figure 3):

$$\phi_{sc} \propto \mu_0 c^2 \frac{Q}{L} \tag{1.5}$$

For harmonic time dependence:

$$I \propto e^{i\omega t} \rightarrow Q \propto \frac{I}{i\omega} \tag{1.6}$$

so the potential scales as:

$$\phi_{sc} \propto -i\mu_0 c^2 \frac{I_0}{\omega L} \tag{1.7}$$

Using the relation $\omega = \frac{2\pi c}{\lambda}$, we have:

$$V_{sc,\phi} \propto i\mu_0 c I_0 \left(\frac{\lambda}{L}\right) = iV_0 \left(\frac{\lambda}{L}\right) \tag{1.8}$$

where $V_0 = \mu_0 c I_0$, the product $\mu_0 c$ being the impedance of free space, 377 Ω, so that V_0 is a sort of characteristic voltage associated with the current I_0. Note that we have used the conventions of figure 1 for the definition of the sign of the voltage and current. A positive electric field (towards increasing *z*, here directed to the right) gives rise to a positive voltage, so since the electric field is -∇ϕ, a negative potential at the right (high *z*) gives rise to a positive voltage.

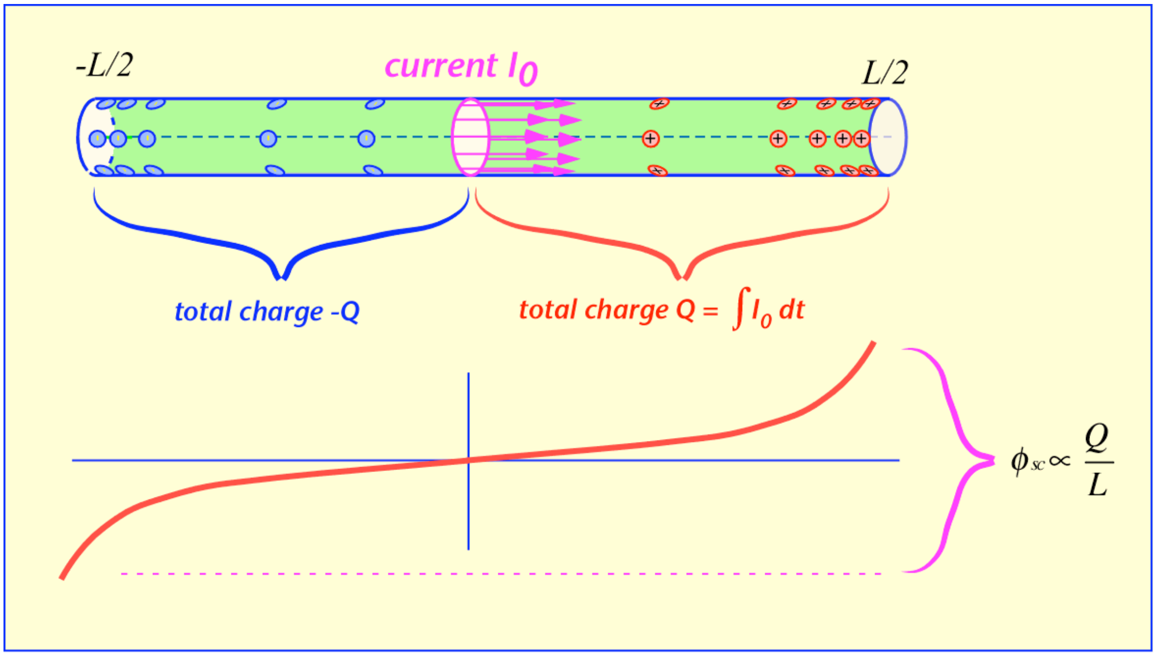

Figure 3: schematic depiction of electrostatic potential along the center of the wire, shown delayed by 1/4 cycle from the current

Physically this equation tells us that at lower frequencies the charge has a longer time to accumulate and thus grows larger in magnitude, and the resulting field is larger when the wire is short and the charges are close together. Thus the electrostatic potential is important at low frequencies and short wire lengths.

The 'instantaneous' or magnetostatic vector potential is taken to be the sum of contributions from all the current elements weighted by (1/distance) from the point of measurement, with any propagation delay ignored. We will choose to find the potential along the axis of the wire, which greatly simplifies the calculation. From the potential version of Ampere's law we have:

$$A_{sc,\,in} = \frac{\mu_0}{4\pi} \iiint \frac{J}{r}\, dv \approx \frac{\mu_0}{4\pi} \int \frac{I}{r}\, dz \qquad (1.9)$$

where the second expression arises from assuming that the currents are localized on the surface of the wire, and uniform around the wire, so that all the current elements at an axial position z are at the same distance from the point of interest. The largest contribution to the integral arises from nearby currents (figure 4), so the potential at any given location along the wire is in phase with and proportional to the current flow, but only weakly dependent on the length of the wire (we shall find the dependence to be logarithmic).

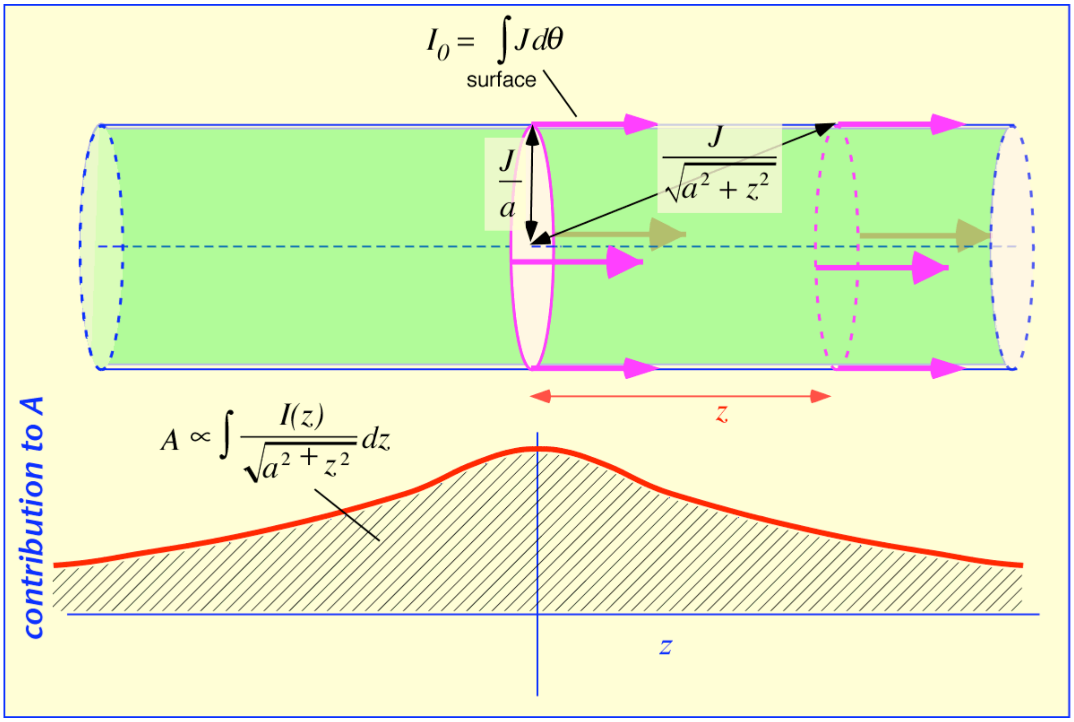

Figure 4: contributions to the instantaneous vector potential along the axis of the wire

The electric field due to this potential thus scales roughly as:

$$E_{sc,\,in} \propto -i\omega\mu_0 I_0 \propto -i\mu_0 c I_0 \frac{1}{\lambda} \tag{1.10}$$

The resulting voltage on the wire is roughly the product of the field and the length:

$$V_{sc,\,in} \propto -i\mu_0 c I_0 \left(\frac{L}{\lambda}\right) = -iV_0 \left(\frac{L}{\lambda}\right) \tag{1.11}$$

where we have employed the characteristic voltage V_0 defined in equation (1.5). By comparison of equations (1.5) and (1.8), we see that the voltages due to the instantaneous charges and current are of opposite sign and scale inversely with respect to the normalized length of the antenna. If, for example, we start near DC and increase the frequency of the impinging radiation, the electrostatic contribution will fall and the magnetostatic contribution will rise (figure 5). It is reasonable to guess, as depicted in the figure, that at some frequency the two might cancel: that is, a ***resonant*** frequency is likely to exist. At resonance, the current generates no net voltage along the wire (in this approximation); in order to cancel the incident voltage and create zero electric field in the wire, we would require an infinite amount of current to flow. In practice, the current does grow large at resonance, but is limited by the effects of the delayed potential, as we shall show in a moment.

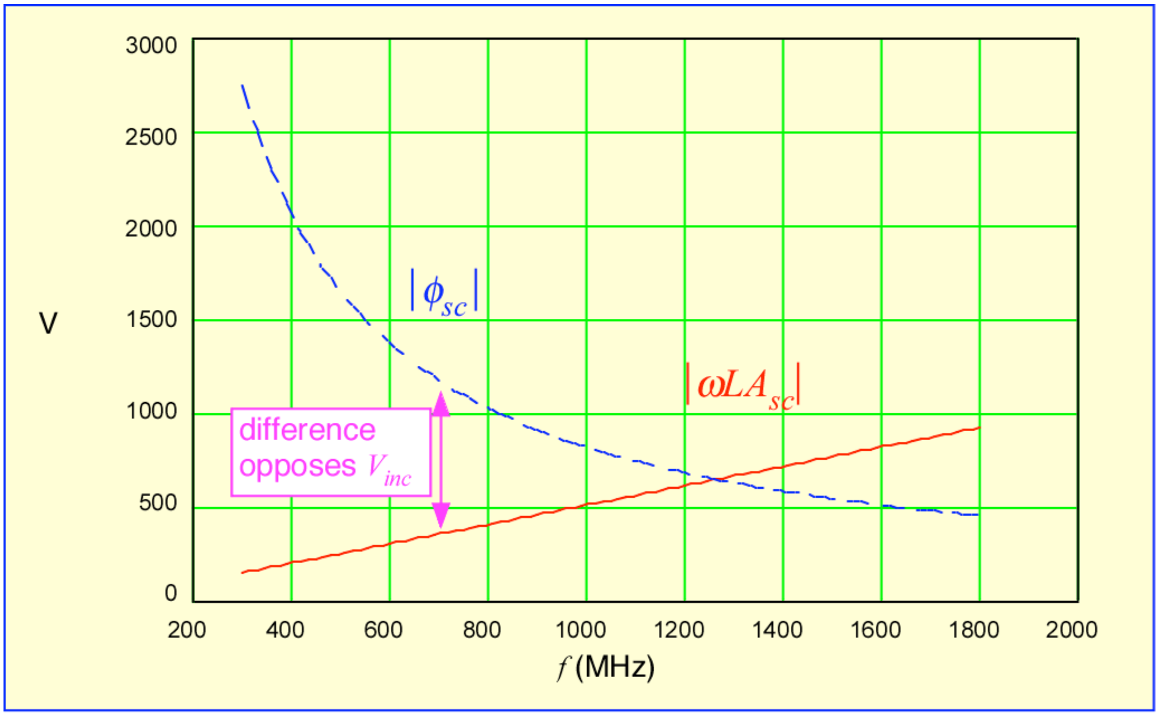

Figure 5: example of electrostatic and magnetostatic contributions to the scattered voltage as a function of frequency; current of 1 ampere, antenna length 0.1 m

Finally, the delayed component is the result of considering the finite speed of light. The potential due to currents from far away is increasingly delayed – for harmonic time dependence, more of that potential is along the delayed (negative imaginary) axis. To first order, this effect cancels the (1/r) decrease in the contribution of more distant currents, so that the contribution of each current element to the delayed potential is independent of position (assuming, of course, that the antenna is short compared to a wavelength). Mathematically, for a harmonic time dependence, the contribution to the potential of any current element must be multiplied by an exponential term e^{-ikr}. When we expand the exponential to first order, ($1-ikr$), we find that we have already accounted for the first term, which is the instantaneous potential. The first-order delayed term, $-ikr$, is linearly dependent on the distance between the current element and the point of measurement; this linear dependence compensates for the inverse weighting of the more distant currents to give a potential integral with no dependence on the distance between the measurement and the currents:

$$A_{sc,\,d} = \frac{\mu_0}{4\pi}\int \frac{J(-ikr)}{r}dv = -ik\frac{\mu_0}{4\pi}\int I dz \tag{1.12}$$

The delayed component is thus linearly proportional to the wavevector $k = 2\pi/\lambda$. If the current doesn't vary too much over the wire, the integral is roughly just the product of the current and the wire length (figure 6). We obtain:

$$A_{sc,\,d} \propto -ik\mu_0 I_0 L \propto -i\mu_0 I_0\left(\frac{L}{\lambda}\right) \tag{1.13}$$

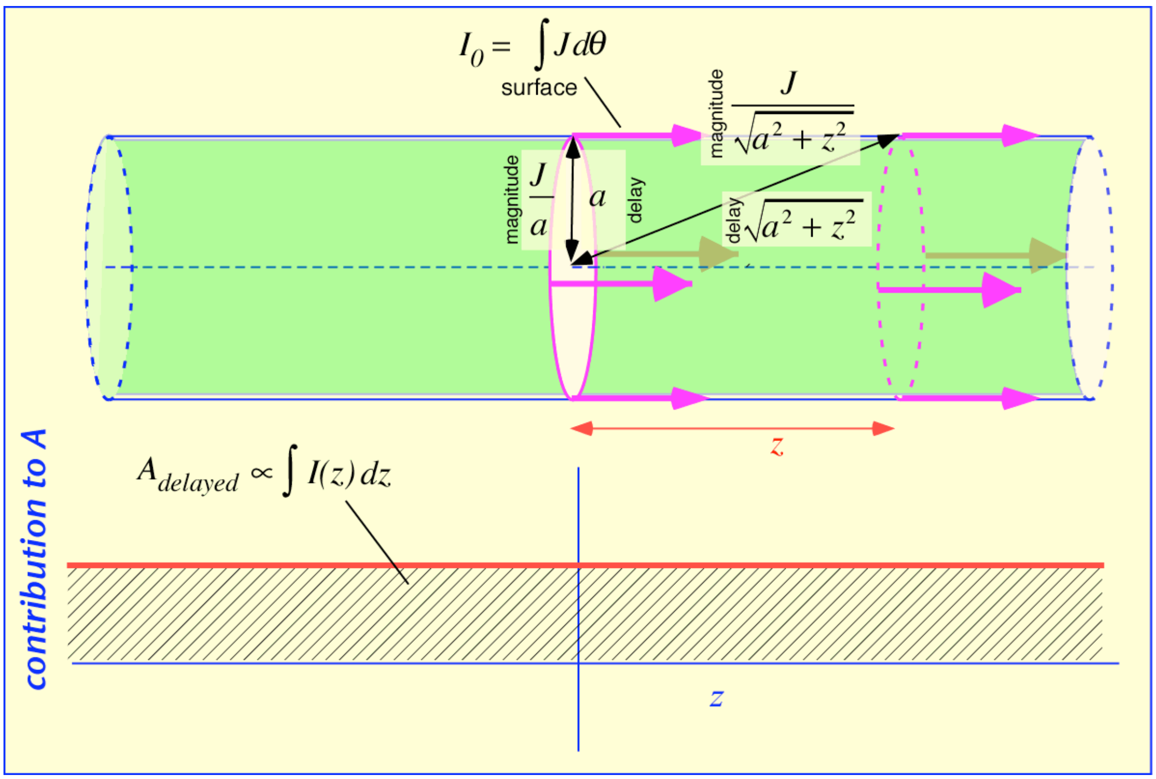

Figure 6: contributions to delayed vector potential along the axis of the wire; as long as the delay is small increasing delay compensates decreasing magnitude, so that all parts of the wire contribute equally.

The induced electric field scales as

$$E_{sc,\,d} \propto -i\omega A_{sc,\,d} \propto -\mu_0 c I_0\left(\frac{L}{\lambda^2}\right) \tag{1.14}$$

and the induced voltage is once again of the product of the field and the wire length:

$$V_{sc,\,d} \propto -V_0 \left(\frac{L}{\lambda} \right)^2 \tag{1.15}$$

Note that the electric field in equation (1.11) is real and opposed to the current. If this is the only contribution to the field, in order to cancel the incident field the current will be in phase with the incident field so that the scattered and incident fields are of opposite sign: that is, the current is flowing in phase with the incident field. The wire is ***acting like a resistor*** even though we have completely ignored the finite conductivity of the wire. We can write $I = V_{inc}/R_{rad}$, where R_{rad} is known as the ***radiation resistance*** of the wire. The energy dissipated by this apparent resistance is radiated away to the distant world, though the value of the resistance is obtained through a purely local calculation[1].

Since the total scattered voltage must be equal in magnitude to the incident voltage, we can write a simple expression for the magnitude of the current:

$$\left| I_0 \right| \propto \frac{\left| V_{inc} \right|}{\mu_0 c \sqrt{\left(\kappa_1 \left(\frac{\lambda}{L} \right) - \kappa_2 \left(\frac{L}{\lambda} \right) \right)^2 + \kappa_3 \left(\frac{L}{\lambda} \right)^4}};$$

$$\text{at resonance } I_0 \propto \frac{\left| V_{inc} \right|}{\mu_0 c} \tag{1.16}$$

where the κ's are as-yet-undetermined constants of order 1. Figure 7 summarizes the relationship between incident voltage, current, and scattered voltage, for various possible values of the scaling parameter (L/λ).

[1] In fact, we have made an important assumption about the outside world, in asserting that only retarded potentials are present. A time-symmetric electrodynamics can be formulated by abandoning this assumption, at the cost of making thermodynamic assertions about the distant universe; see Mead [4] and references therein.

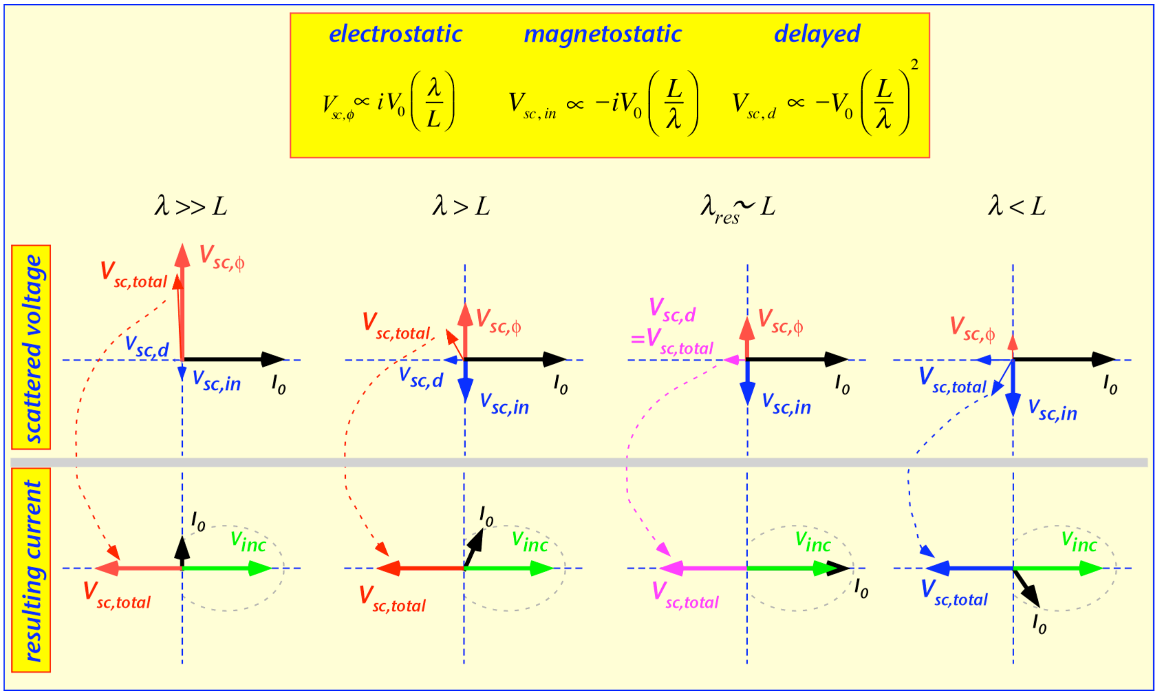

Figure 7: phase and amplitude of scattered voltages relative to induced current (top row) and incident voltage (bottom row) for various values of normalized length

At long wavelengths (low frequencies) the scattered voltage is dominated by the strong electrostatic contribution. The current must lead the incident voltage by 90 degrees, but only a small current is needed. With increasing frequency, wavelength becomes only modestly longer than the wire. The electrostatic voltage contribution is partially cancelled by the magnetostatic voltage, and the delayed component becomes significant; the overall phase of the scattered voltage is larger, and thus the current moves closer to the real axis. At resonance, the electrostatic and magnetostatic contributions cancel, leaving only the small delayed voltage. A large current, in phase with the incident voltage, is required to cancel the incident voltage. At still higher frequencies and smaller wavelengths, the magnetostatic contribution begins to dominate the scattered voltage, and the current begins to lag the voltage. We can only proceed a short ways past resonance before more sophisticated approximations are needed, as the wire is no longer short compared to a wavelength. Note that in this discussion we have suppressed all the constant terms in the interests of simplicity. In fact, we will find that resonance occurs when the wavelength is a bit more than twice the length of the wire.

So far we have examined the scattered potentials and fields along the axis of the wire. However, the source equations apply everywhere. At large distances from the wire, the current on the antenna gives rise to a potential delayed by the speed of light; for a harmonic disturbance the delay simply introduces a phase shift. Perpendicular to the axis of the wire, the distance between a test point and any point on the wire is equal when the distance is large. Thus the vector potential is simply the integral of current over the length of the wire – the same integral that determines the delayed constituent of the local magnetic potential – divided by the distance (equation 1.16, figure 8):

$$A_{sc,\,far} \propto \mu_0 \frac{e^{-ikr}}{r} \int I dz \approx \mu_0 \frac{e^{-ikr}}{r} I_0 L \qquad (1.17)$$

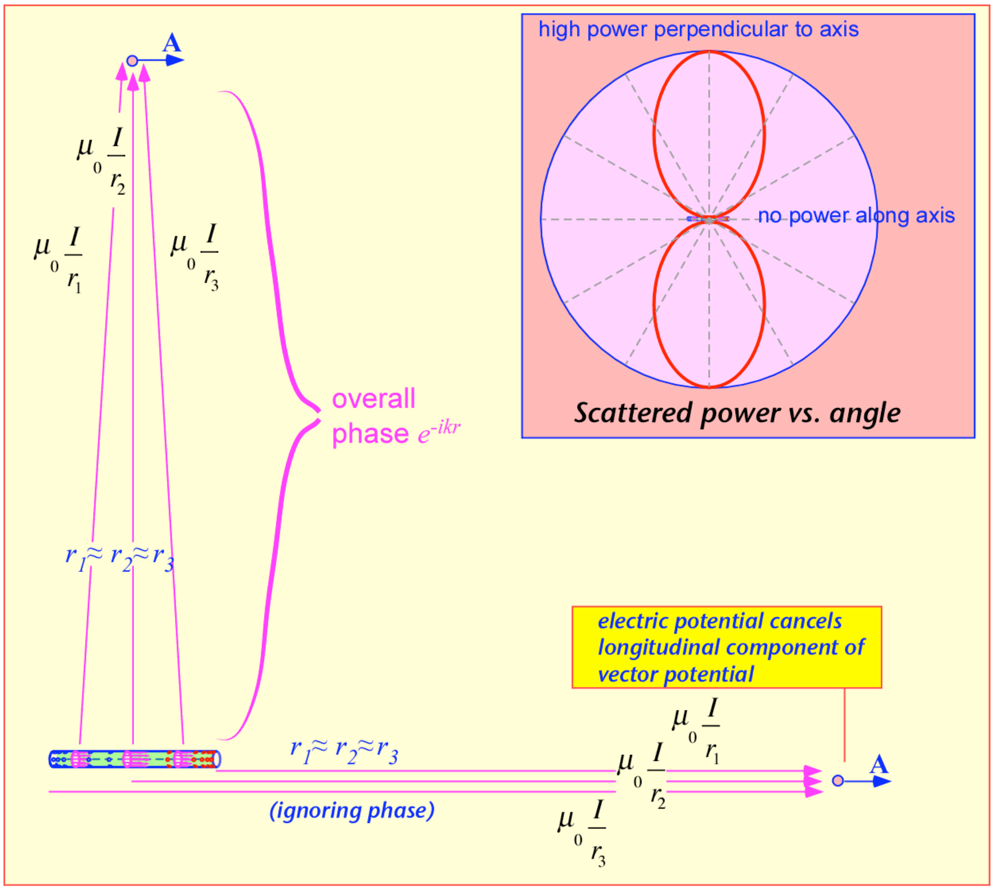

Figure 8: scattered potential at long distances is the sum of contributions from current along the wire; only the transverse component (perpendicular to r) contributes to net power transfer

Since the current is purely along the axis of the wire, the vector potential everywhere is directed parallel to the wire; but it can be shown that only that part of the potential perpendicular to the direction of propagation couples with distant charges or currents, the longitudinal component being cancelled by the electric potential contribution [5]. Projection of the potential perpendicular to the vector **r** introduces a factor of sin(θ) where θ is the angle between **r** and the axis of the wire. (A complete calculation would obtain a more complex angular dependence, due to the variation in distance and thus phase for current elements at differing locations on the wire, but the distinction is of minor import for short antennas.) The wire scatters part of the incident energy, predominantly in the plane perpendicular to the wire axis (figure 8, inset). Since the amount of scattering is proportional to the current, the scattering cross-section is maximized at the resonant frequency.

We can also estimate the total scattered power from the wire. The characteristic voltage V_0 is the product of the current and the impedance of free space, $\mu_0 c = 377\ \Omega$, so the current at resonance is of the order of the incident voltage divided by $\mu_0 c$ (equation 1.16).This current in turn produces a scattered potential in the far field:

$$A_{sc} \propto \frac{\mu_0 I_0 L}{r} = \frac{\mu_0 L}{r}\frac{V_{inc}}{\mu_0 c} = \frac{L}{r}\frac{V_{inc}}{c} \approx \frac{L}{r}\frac{\omega A_{inc} L}{c} \qquad (1.18)$$

The radiated power density has the form of the square of the local electric field divided by the impedance of free space (that is, V^2/R per unit area):

$$U \propto \frac{|\omega A|^2}{\mu_0 c} = \frac{\omega^2}{\mu_0 c}\left(\frac{\omega A_{inc} L^2}{rc}\right)^2 = \underbrace{\frac{\omega^2 A_{inc}{}^2 L^2}{\mu_0 c}}_{P_{inc}}\frac{\omega^2 L^2}{r^2 c^2} \qquad (1.19)$$

Note that we have written the radiated power density in terms of a quantity P_{inc}, which is simply the incident power on a square region L on a side. Ignoring the angular dependence (which introduces a constant factor of order 1) the total radiated power scales as:

$$P \propto Ur^2 = \underbrace{\frac{\omega^2 A_{inc}{}^2 L^2}{\mu_0 c}}_{P_{inc}}\frac{\omega^2 L^2}{c^2} \propto P_{inc}\left(\frac{L}{\lambda}\right)^2 \approx P_{inc} \text{ at resonance} \qquad (1.20)$$

That is, to within a constant factor, the amount of power scattered by a wire at resonance is equal to the power that falls on a square region of area about L^2: the scattering cross-section of the wire is quite large even if the wire is extremely thin.

Let's get a feeling for the size of the various quantities we've introduced. Remember that these are only order-of-magnitude values since we have not yet attempted to derive the constant terms and parameter dependencies. An impinging voltage of 1 V produces a current on the order of (1/377) ≈ 2.5 mA, for a resonant antenna on the order of a wavelength long. If the length of the antenna is (say) 0.1 meter, the electric field is about 10 V/m and the scattered power is around (100/377)(0.01) or about 3 mW. A short wire (say on the order of $\lambda/10$) will have a capacitive current around (1/10) of this value, and a real current (in phase with the impinging field) of perhaps (1/100).

Note that since the real current scales as the second power of the wavelength, the scattered power (which goes as the square of the current) scales as the ***fourth power of the ratio of antenna size to wavelength***. This may be a familiar result: it is known as ***Rayleigh scattering***, and with antennas the size of individual molecules explains why the cloudless sky is blue.

2. Wires and Antennas

In order to use a wire as a means to convert between voltages and waves – as an ***antenna*** – we need to have a method of making a connection. One common approach is to break the wire in the center and connect one lead of a balanced transmission line to each end: a ***dipole antenna***, as depicted in figure 9.

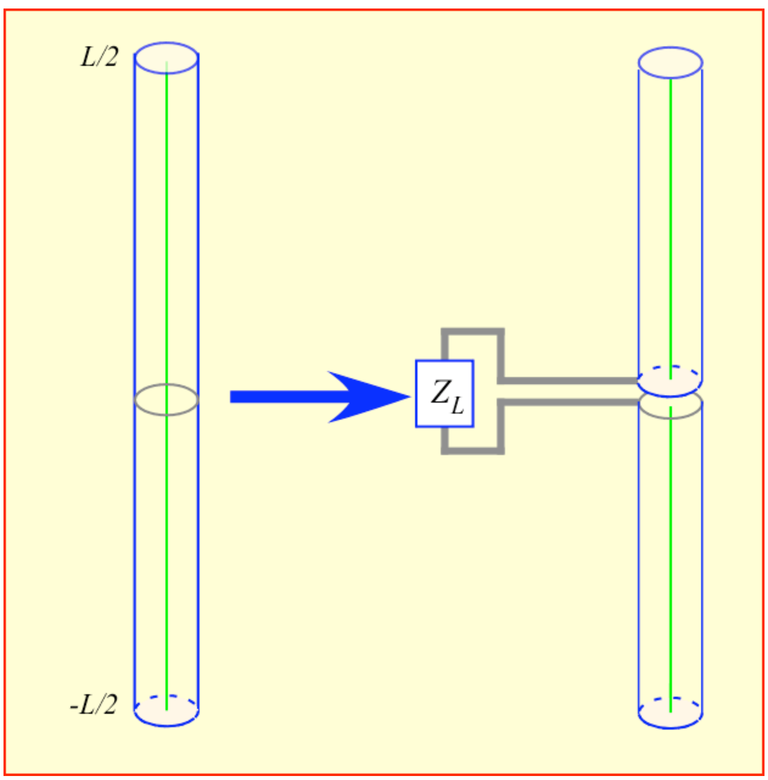

Figure 9: transmission line connected to the two halves of a wire to form a dipole

When the load impedance is 0, the ***short-circuit current*** flowing in the dipole is very similar to the current discussed above in the wire. When the load impedance is infinite, the ***open-circuit voltage*** is nearly equal to the voltage across a single wire half the length of the dipole; for antennas much shorter than a wavelength this is approximately $E_{inc}L/2$.

The behavior of the antenna for any load can then be inferred from the open-circuit voltage and short-circuit current using Thévenin's theorem (figure 10).

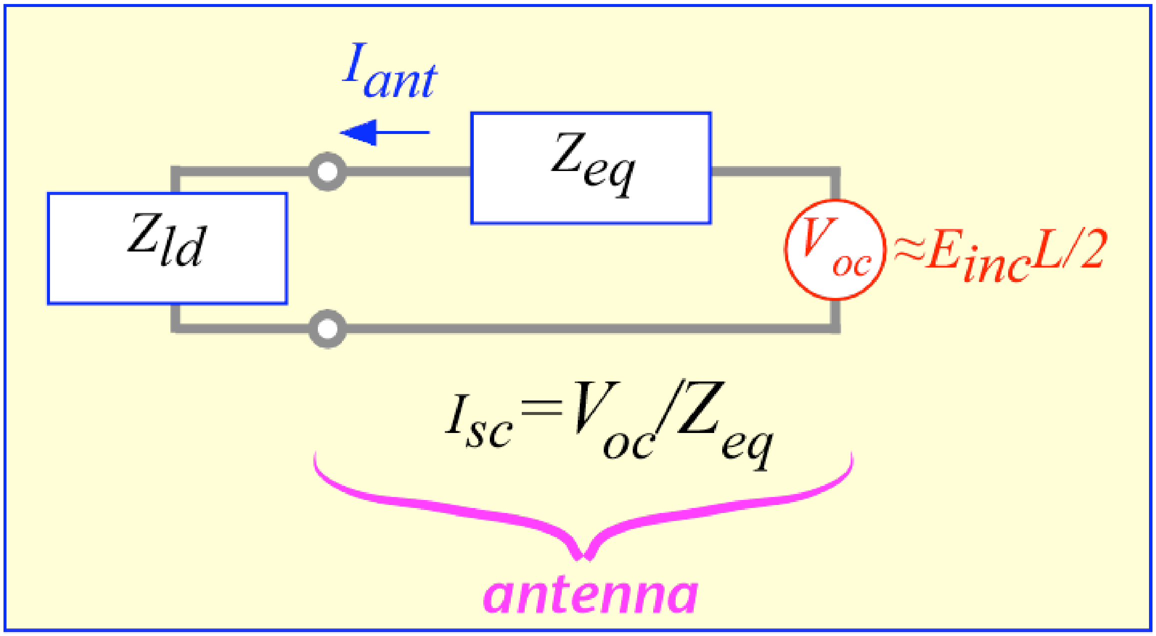

Figure 10: Thevenin equivalent circuit for dipole antenna; short-circuit current is I_0 of a wire of length L; open-circuit voltage is the voltage across a wire of length L/2.

The current distribution for any load is the superimposition of the current distributions corresponding to the open-circuit and short-circuit conditions:

$$I_{total} = \alpha I_{oc} + \beta I_{sc}$$
$$Z_{LD} = \left(\frac{V}{I}\right)_{\text{terminals}} = \frac{\alpha V_{oc}}{\beta I_{sc}} = \frac{\alpha}{\beta} Z_{eq} \qquad (1.21)$$

At low frequencies relative to resonance, the source impedance is almost purely imaginary. A conjugate-matched load will be pure imaginary of opposite sign (that is, a resonant inductor for the capacitance of the antenna). The current distribution will therefore be the difference of the long and short currents:

$$I_{total} = I_{oc} - I_{sc} = I_0\left(1 - \left[\frac{2z}{L}\right]^2\right) - I_{02}\left(1 - \left[\frac{4z'}{L}\right]^2\right) \text{ where } z' = z \mp \frac{L}{4} \qquad (1.22)$$

The full wire contributes a quadratic term $4z^2$ and the short segments contribute quadratic terms four times as large. For antennas short compared to resonance, the peak current is proportional to the product of the incident voltage and the length, but the incident voltage is $E_{inc}L$, so the current is proportional to the square of the length. Thus $I_{02} = I_0 / 4$: the quadratic terms cancel, and the current is linear in position. Around resonance, the current into a matched load is dominated by the large current flowing on the full length wire, and is close to quadratic along the length of the antenna. At twice the resonant frequency, the individual segments are near resonance and support a large current (producing a large open-circuit voltage), while the short-circuit current is modest away from resonance; as a result, the equivalent resistance is large and the current distribution nears the open-circuit case.

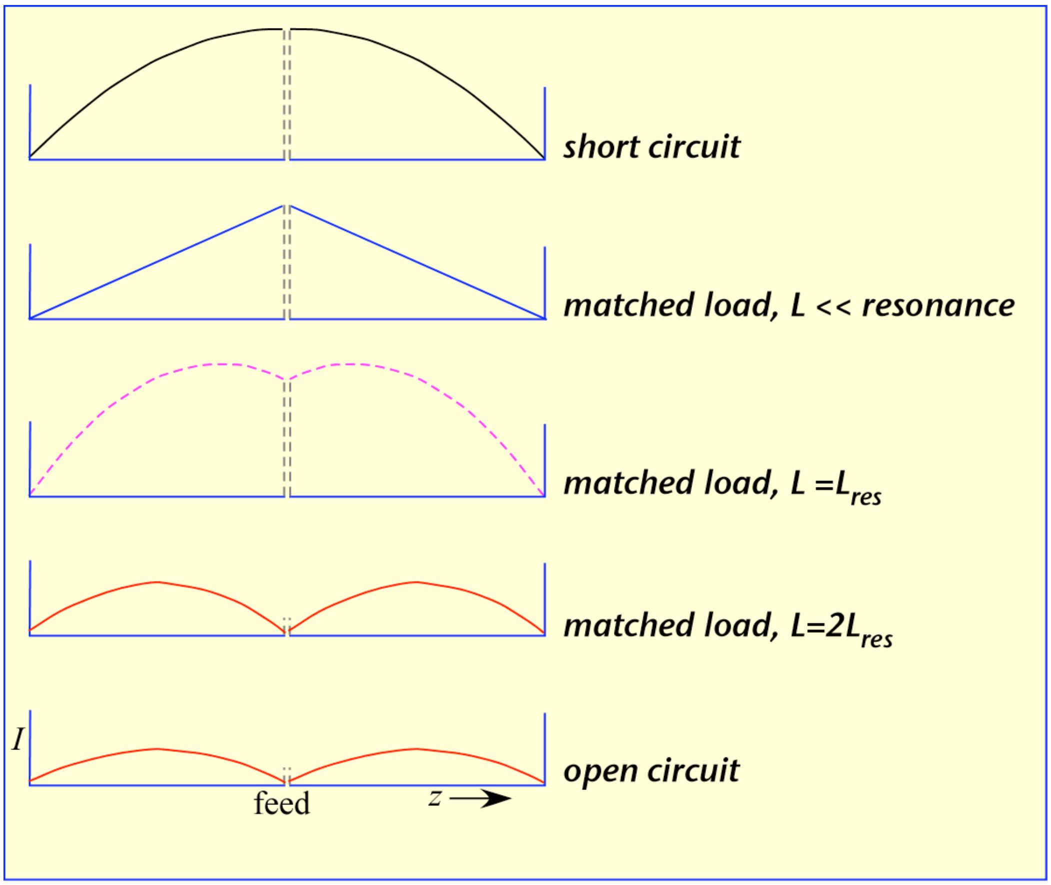

Figure 11: current distributions on a dipole antenna constructed from the open- and short-circuit quadratic distributions

The power delivered to a matched load is proportional to $V_{oc}^2 / \mathrm{Re}(Z_{eq})$. At resonance the load is real and scales as the impedance of free space; thus the power delivered to the load has the same form as the scattered power (equation 1.19), and it is reasonable to infer that the two quantities are of similar magnitude: at resonance the antenna collects power from an ***effective area*** roughly equivalent to a square L on a side. For longer wavelengths or lower frequencies, the real part of the impedance falls as the square of the ratio of length to wavelength (equation 1.15), but the open-circuit voltage also falls linearly, so the power delivered to a matched load is ***independent of the size of the***

antenna. (Note that in practice the load required becomes difficult to fabricate for very short antennas.)

A more precise treatment must incorporate the interaction of the two halves of the dipole.

To summarize the discussion so far, we have shown by heuristic arguments that current flow in a short wire ought to lead the incident voltage by 90 degrees (like a capacitive load), but that at some frequency where the wire is roughly a wavelength long, the current should be large and in phase with the incident voltage. At higher frequencies still, the current lags the voltage, as in an inductor. The portion of the current in phase with the incident voltage corresponds to power scattered away from the wire, mainly in the plane perpendicular to the wire axis. By splitting the wire at its center we obtain a dipole antenna, whose behavior as a receiver can be very simply estimated using the results we obtained for a continuous wire. As promised, all these results have been obtained without the need to find the magnetic field **B**.

It is important to note that these results apply only to antennas of total length comparable to or less than a wavelength; for longer antennas it is no longer tenable to approximate the phase as a linear function of distance, and contributions to the 'instantaneous' potentials from distant parts of the antenna change sign.

In part II of this article we will exhibit the detailed mathematics for implementing the calculations sketched out in part I, focusing on a simple quadratic approximation to the current distribution on the wire.

Part II: Detailed Estimates of Scattered Voltage and Current

1. Doing the Integrals

We will now fill in the details of the calculation of scattered current and voltage for a short length of wire with an impinging potential (figure 1). Let us review briefly the procedure we shall follow:

- We first guess at a current distribution along the wire
- We calculate the resulting scattered electrostatic and magnetic potentials using equations (1.1) and (1.2)
- We find the electric field using (1.3), and integrate along the wire to find the scattered voltage due to each constituent of the scattered potentials
- We adjust the magnitude and phase of the scattered current to cancel as nearly as possible the incident voltage

We'll start with a guess that the current is quadratic with position (figure 11). A quadratic current distribution has several desirable properties: it is smooth, goes to zero at the ends, and promises to be (relatively) easy to integrate analytically. The resulting charge distribution, being the derivative of the current with position, is linear, with zero charge at the center of the wire.

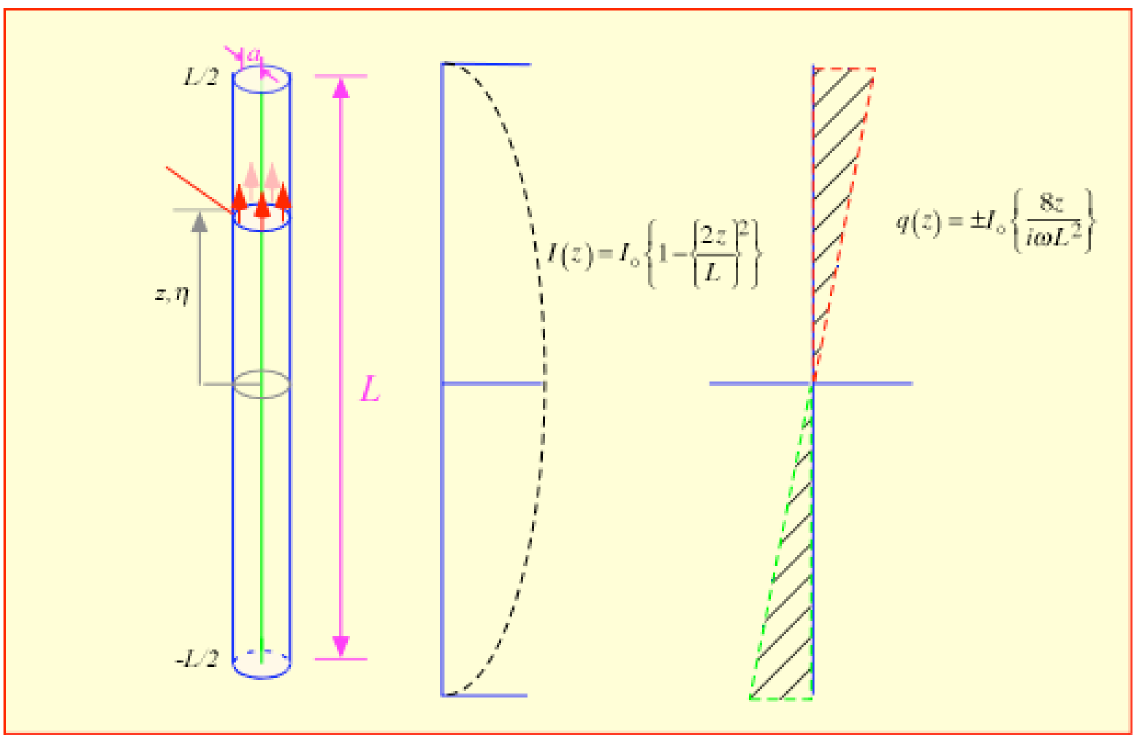

Figure 12: quadratic current distribution with maximum value I_0 at the center of the wire, and linear charge distribution with maximum magnitude $4I_0/\omega L$, for a wire of length L

We now proceed calculate the resulting vector and scalar potentials by integrating over the distributions of current and charge. In performing the integrals analytically it is useful to be aware of a couple of results we shall use repeatedly:

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right) \tag{2.1}$$

and

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} \tag{2.2}$$

The electric (scalar) potential at a location η along the wire is found by integrating over the charge distribution (as a function of position *z*) from figure 12:

$$\begin{aligned}\phi(\eta) &= \frac{\mu_0 c^2}{4\pi} \int_{-L/2}^{L/2} \frac{q(z)\,dz}{\sqrt{a^2 + (z-\eta)^2}} = \frac{\mu_0 c^2}{4\pi} \frac{8I_0}{i\omega L^2} \int_{-L/2}^{L/2} \frac{z dz}{\sqrt{a^2 + (z-\eta)^2}} \\ &= \frac{\mu_0 c^2}{4\pi} \frac{8I_0}{i\omega L^2} \int_{-L/2-\eta}^{L/2-\eta} \frac{(u+\eta)\,du}{\sqrt{a^2+u^2}} \quad \text{where } u = z - \eta\end{aligned} \tag{2.3}$$

The bottom version of the integral is the sum of two integrals, one of the form of equation (2.1) and the second like (2.2), so we can use those results to perform the integrations:

$$\begin{aligned} &= \frac{\mu_0 c^2}{4\pi} \frac{8I_0}{i\omega L^2} \left\{ \int_{-L/2-\eta}^{L/2-\eta} \frac{u du}{\sqrt{a^2+u^2}} + \eta \int_{-L/2-\eta}^{L/2-\eta} \frac{du}{\sqrt{a^2+u^2}} \right\} \\ &= \frac{\mu_0 c^2}{4\pi} \frac{8I_0}{i\omega L^2} \left\{ \sqrt{a^2+u^2}\,\Big|_{-L/2-\eta}^{L/2-\eta} + \ln\left(u + \sqrt{a^2+u^2}\right)\Big|_{-L/2-\eta}^{L/2-\eta} \right\} \\ &= \frac{\mu_0 c^2}{4\pi} \frac{8I_0}{i\omega L^2} \left\{ \begin{array}{l} \sqrt{a^2 + (L/2-\eta)^2} - \sqrt{a^2 + (L/2+\eta)^2} + \\ \eta \ln\left(\dfrac{L/2 - \eta + \sqrt{a^2 + (L/2-\eta)^2}}{-L/2 - \eta + \sqrt{a^2 + (L/2+\eta)^2}} \right) \end{array} \right\} \end{aligned} \tag{2.4}$$

This result looks rather intimidating, but recall that in most cases of interest, the diameter of the wire is much smaller than the length (that is, the wire is thin). Using this fact, we can Taylor-expand the result of equation (2.4) as long as we are a few diameters away from the ends of the wire, obtaining a much simpler expression:

$$\phi(\eta) \approx \frac{\mu_0 c^2}{4\pi} \frac{8 I_0}{i\omega L^2} \eta \left\{ \ln\left(\frac{(L/2)^2 - (\eta)^2}{a^2} \right) + \ln(2) - 2 \right\} \quad (2.5)$$

Using $\omega = \frac{2\pi c}{\lambda}$, we can reformulate the potential in terms of the characteristic voltage V_0:

$$\phi(\eta) \approx \mu_0 c I_0 \frac{\lambda \eta}{i\pi^2 L^2} \left\{ \ln\left(\frac{(L/2)^2 - (\eta)^2}{a^2} \right) + \ln(2) - 2 \right\}$$

$$= \frac{V_0}{i\pi^2} \left(\frac{\lambda}{L} \right) \left(\frac{\eta}{L} \right) \left\{ \ln\left(\frac{(L/2)^2 - (\eta)^2}{a^2} \right) + \ln(2) - 2 \right\} \quad (2.6)$$

At the wire ends, the expression is of the form ln($(L/2)^2 - (L/2)^2$) = ln 0 $\Rightarrow\infty$ (albeit very weakly): that is, the Taylor expansion is not valid. The correct limiting value at the ends is:

$$\phi\left(\pm \frac{L}{2} \right) = \mp i \frac{\mu_0 c^2}{4\pi} \frac{8 I_0}{\omega L} \left\{ \frac{1}{2} \ln\left(\frac{2L}{a} \right) - 1 \right\}$$

$$= \mp i \frac{V_0}{\pi^2} \left(\frac{\lambda}{L} \right) \left\{ \frac{1}{2} \ln\left(\frac{2L}{a} \right) - 1 \right\} \quad (2.7)$$

An example of the potential distribution, and electric fields, for a wire antenna 0.1 meter long and 0.001 m in diameter, with a current of 1 ampere, is shown in figure 12, for three different values of the ratio of length to wavelength. The potential is nearly linear save for the region very near the ends; the electric field is similarly almost constant except near the ends of the wire.

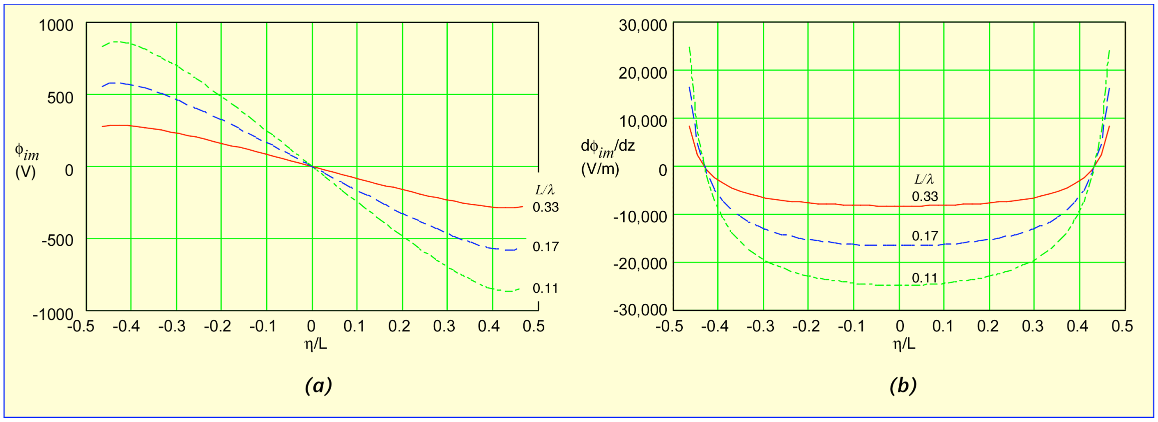

Figure 13: (a) Imaginary part of scalar potential vs. position (b) imaginary part of the spatial gradient of the scalar potential vs. position, for three different values of the ratio (L/λ).

A minor problem arises when we attempt to extract the appropriate scattered voltage: the field is strongly varying very close to the ends of the wire. This tells us that we have not quite arrived at a correct current distribution: we are short some charge at the ends. In the interests of simplicity, we shall avoid trying to solve the problem at this time, and instead fudge it by linearly extrapolating the voltage to the ends. (The amount of charge involved in fixing the problem is tiny, and affects only the potential near the ends, so we don't lose much by this minor fudge.) We then need to choose how close to the ends we should go. It turns out that this choice modestly affects the predicted resonant wavelength / frequency; we have traded a bit of simplicity for an extra ambiguity, that could be resolved with a more precise estimate of the current distribution. In this case we choose to extrapolate from the points at which the field changes sign; in the appendix we show that to a good approximation these locations are at:

$$\eta_{\max} \equiv \eta_{E\to 0} \approx \frac{1}{\ln\left(\frac{L}{a}\right)^2 - 2} \tag{2.8}$$

Since we have defined the voltage so that $z = L/2$ is the ground reference, the induced voltage is the potential at $-L/2$ with respect to that at $L/2$. We calculate the voltage between the points at which the scattered electrostatic field goes to zero, and linearly extrapolate to the ends (though the effects of the extrapolation will be small). We obtain:

$$
\begin{aligned}
V_{sc,\phi} &= -2\phi_{sc}\left(\eta_{\max}\right) \\
&\approx i\frac{V_0}{\pi^2}\left(\frac{\lambda}{L}\right)\left\{2\ln\left(\frac{L}{a}\right) - \ln\left(2\left[\ln\left(\frac{L}{a}\right)-1\right]\right) - 2\right\} \\
&\equiv iV_0\left(\frac{\lambda}{L}\right)\kappa_1
\end{aligned}
\tag{2.9}
$$

This equation has the same form as equation (1.8), with a constant factor ($1/\pi^2$) and a logarithmic term providing the expected weak additional dependence on wire length. For the example we've been using, where (L/a) = 100, the logarithmic term is about (4.6+1) = 5.6.

The vector potential is found by integrating over the currents (equation (2.9)):

$$
\begin{aligned}
A_{sc}(\eta) &= \frac{\mu_0}{4\pi}\int_{-L/2}^{L/2}\frac{I(z)\,dz}{\sqrt{a^2+(z-\eta)^2}} = \frac{\mu_0 I_0}{4\pi}\int_{-L/2}^{L/2}\frac{\left(1-\left(\frac{2z}{L}\right)^2\right)dz}{\sqrt{a^2+(z-\eta)^2}} \\
&= \frac{\mu_0 I_0}{4\pi}\int_{-L/2-\eta}^{L/2-\eta}\frac{\left(1-\left(\frac{2u+2\eta}{L}\right)^2\right)du}{\sqrt{a^2+u^2}} \qquad \text{where } u = z-\eta
\end{aligned}
\tag{2.10}
$$

By expanding the numerator we obtain three integrals, one each of the form of (2.1) and (2.2), and one with a quadratic term u^2 in the numerator, which can be reduced to the form of (2.1) using integration by parts. Evaluation of the integral is straightforward but laborious, and the details are deferred to the appendix. The result is:

$$A_{sc} = \frac{\mu_0 I_0}{4\pi}\left\{ \begin{array}{l} \left(1-\frac{4}{L^2}\eta^2-\frac{2}{L^2}a^2\right)\ln\left(\frac{L/2-\eta+\sqrt{a^2+(L/2-\eta)^2}}{-L/2-\eta+\sqrt{a^2+(L/2+\eta)^2}}\right) \\ +\left[\left(\frac{6}{L^2}\eta-\frac{1}{L}\right)\sqrt{a^2+(L/2+\eta)^2}-\left(\frac{6}{L^2}\eta+\frac{1}{L}\right)\sqrt{a^2+(L/2-\eta)^2}\right] \end{array} \right\} \quad (2.11)$$

This expression is somewhat awkward to work with, and Taylor-expanding still leaves a bit of a mess. However, we can obtain a very good approximation to the answer by fitting a quadratic expression to the exact values of (2.10) at the center and ends of the wire, which are easily found. The potential at the center is:

$$\eta = 0: \quad A_{sc} \approx \frac{\mu_0 I_0}{4\pi}\left\{\ln\left(\left(\frac{L}{a}\right)^2\right)-1\right\} \quad (a << L) \qquad (2.12)$$

and at the ends:

$$\eta = \frac{L}{2}: \qquad A_{sc} \approx \frac{\mu_0 I_0}{4\pi}\{2\} \quad (a << L) \qquad (2.13)$$

The quadratic fit is readily found:

$$\mathrm{A}_{\mathrm{sc}} \approx \frac{\mu_0 I_0}{4\pi}\left\{\left[\ln\left(\left(\frac{L}{a}\right)^2\right)-3\right]\left[1-\left(\frac{2\eta}{L}\right)^2\right]+2\right\} \qquad (2.14)$$

A comparison between the approximate and exact expressions is shown in figure 14. To a very good approximation, the vector potential just looks like the current distribution, but with a constant added everywhere to account for the fact that the potential doesn't go to 0 at the ends. The approximation (2.13) is much easier to integrate analytically than the full expression (2.10) or a Taylor-expanded approximation to it.

Figure 15 shows the product of the vector potential and angular frequency – that is, the magnitude of the electric field – as a function of position for the same three frequencies employed above. The shape of A is independent of frequency; the magnitude of the contribution to electric field is thus linear in frequency.

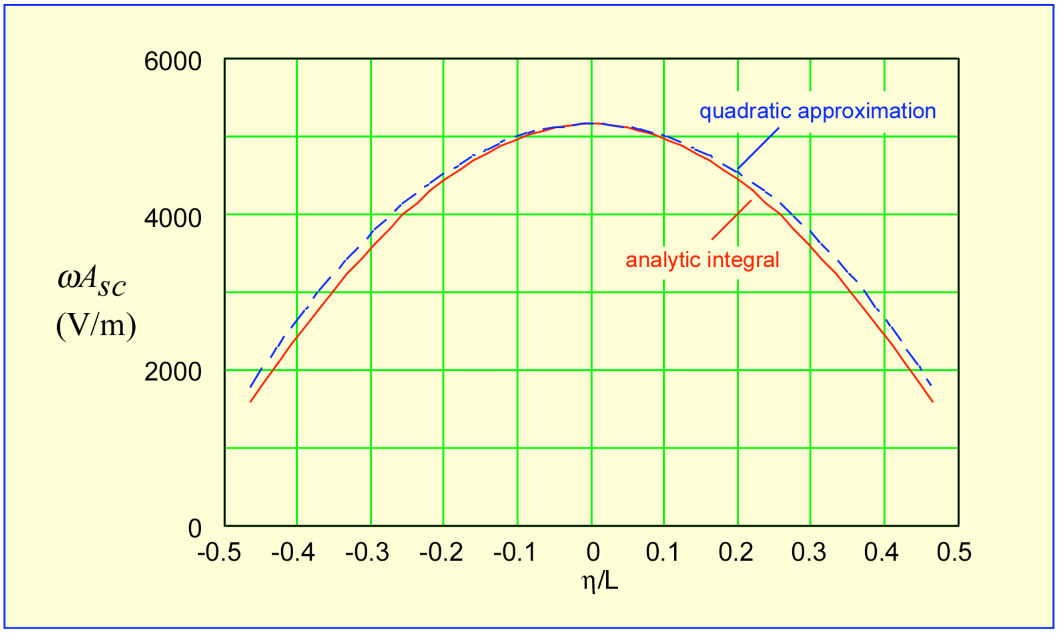

Figure 14: vector potential, multiplied by angular frequency; comparison of quadratic approximation (1.24) to analytic formula (1.21). Parameters of figure 12 with (L/λ) = 0.33.

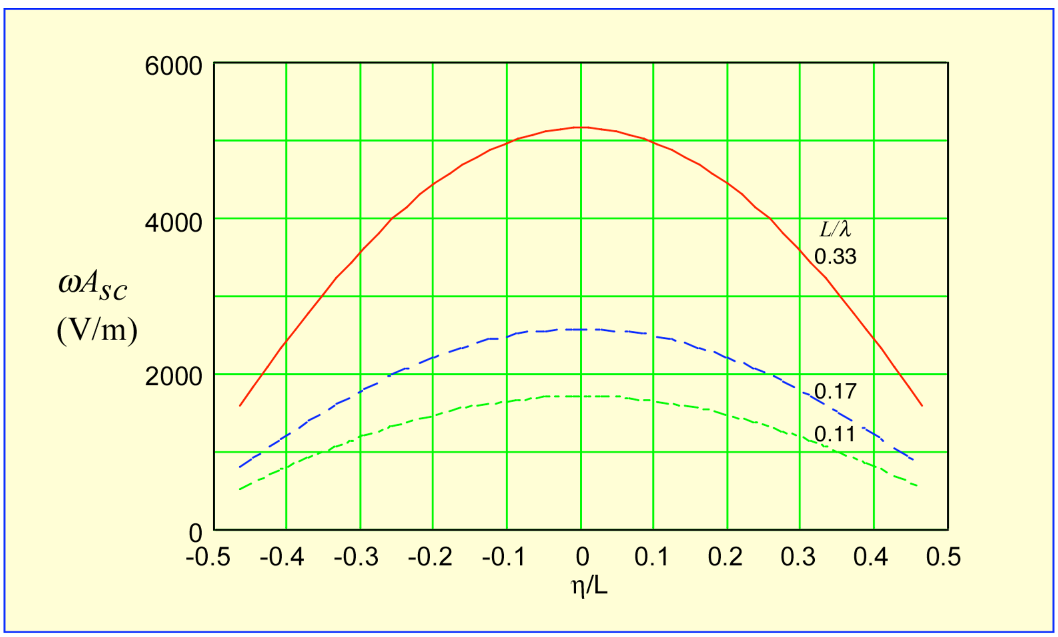

Figure 15: product of vector potential and angular frequency for three frequencies

The dependence of the two contributions to the electric field on frequency is shown in figure 16. The contribution from the scalar (electric) potential decreases with increasing frequency, while the contribution from the vector (magnetic) potential increases linearly with frequency. The two quantities are roughly equal at about 1.3 GHz: the resonant frequency for the example wire.

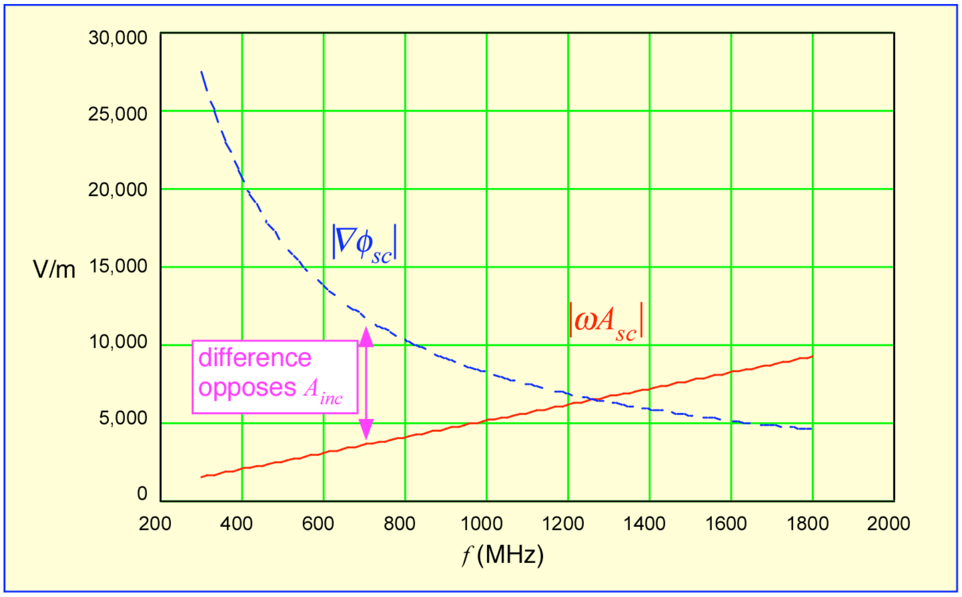

Figure 16: Constituents of the electric field evaluated at the center of the wire as a function of frequency; for L=0.1 m, a=0.001 m

The associated voltage is obtained by integrating the electric field over the length of the wire; after a bit of straightforward labor we obtain:

$$-i\omega\int_{-L/2}^{L/2} \mathrm{A}_{sc}\,dz \approx -i\omega\frac{\mu_0 I_0}{4\pi}\int_{-L/2}^{L/2}\left\{\left[\ln\left(\left(\frac{L}{a}\right)^2\right)-3\right]\left[1-\left(\frac{2z}{L}\right)^2\right]+2\right\}dz$$
$$= -iV_0\frac{L}{\lambda}\left(\frac{2}{3}\left[\ln\left(\frac{L}{a}\right)\right]\right) \equiv -iV_0\frac{L}{\lambda}\kappa_2 \qquad (2.15)$$

This expression is of the same form as equation (1.11), but now we see that there is a constant of order 1 and a logarithmic term in the ratio of wire length to radius.

We now have two contributions to the voltage. The third contribution is the delayed component of the vector potential, due to the finite velocity of light. Recall that, if the antenna is short enough to allow a first-order expansion of the exponential, the vector potential integral takes on the very simple form (equation (1.12), repeated below for convenience as (2.14)):

$$A_{sc,d} = -ik\frac{\mu_0}{4\pi}\int I dz \tag{2.16}$$

It is easy to show that for a quadratic current distribution, the average value is 2/3 of the peak value, so we have:

$$\begin{aligned} A_{sc,d} &\approx -ik\frac{\mu_0 I_0}{4\pi}\left(\frac{2L}{3}\right) = -i\frac{\mu_0 I_0}{3}\left(\frac{L}{\lambda}\right) \\ \rightarrow -i\omega A_{sc,d} &= -\frac{2\pi}{3}V_0\left(\frac{L}{\lambda^2}\right) \end{aligned} \tag{2.17}$$

Now, while the vector potential is the largest contribution to the real part of the electric field, when we try to get semi-quantitative results we must also take into account the scalar potential. The first-order term is just the average of the charge, and vanishes identically since charge is presumed conserved on the wire. However, the third-order term (from the expansion of the exponential) turns out to be of comparable magnitude to the scalar potential term:

$$\begin{aligned} &\phi_{sc,d[3]}(\eta) = i\frac{\mu_0 c^2}{4\pi}\left(\frac{8I_0}{i\omega L^2}\right)\left(\frac{k^3}{6}\right)\int_{-L/2}^{L/2} z\left((z-\eta)^2 + a^2\right)dz \\ &= -\frac{2\pi}{9}V_0\left(\frac{L}{\lambda^2}\right)\eta \rightarrow E_{sc,d[\phi]} = \frac{2\pi}{9}V_0\left(\frac{L}{\lambda^2}\right) \end{aligned} \tag{2.18}$$

The electric field is the sum of the two contributions, and is constant with position in this approximation:

$$\begin{aligned} &E_{sc,d} = -i\omega A_{sc,d} + E_{sc,d[\phi]} = -\frac{2\pi V_0}{3}\left(\frac{L}{\lambda^2}\right) + \frac{2\pi V_0}{9}\left(\frac{L}{\lambda^2}\right) \\ &= -\frac{4\pi V_0}{9}\left(\frac{L}{\lambda^2}\right) \end{aligned} \tag{2.19}$$

Thus the integral is just the product of the field and the length:

$$V_{sc,d} = -\frac{4\pi V_0}{9}\left(\frac{L}{\lambda}\right)^2 \tag{2.20}$$

Note that this expression is of the same form as equation (1.15), with a constant factor (2π/3).

The total scattered voltage for a given current flow I_0 and corresponding characteristic voltage V_0 is the sum of the three components, equations (2.8), (2.14), and (2.18):

$$\begin{aligned} V_{sc} &= V_{sc,\phi} + V_{sc,in} + V_{sc,d} \\ &= iV_0\left\{\kappa_1\left(\frac{\lambda}{L}\right) - \kappa_2\left(\frac{L}{\lambda}\right)\right\} - \frac{4\pi V_0}{9}\left(\frac{L}{\lambda}\right)^2 \end{aligned} \tag{2.21}$$

where κ_1 was defined in equation (2.9) above. The incident and scattered voltages should add to 0:

$$\begin{aligned} &V_{inc} + V_{sc} = 0 \\ &= V_{inc} + iV_0\left\{\kappa_1\left(\frac{\lambda}{L}\right) - \kappa_2\left(\frac{L}{\lambda}\right)\right\} - \frac{4\pi V_0}{9}\left(\frac{L}{\lambda}\right)^2 \end{aligned} \tag{2.22}$$

where $V_{inc} = E_{inc}L = -i\omega A_{inc}L$. Solving for V_0 we obtain:

$$V_0 = \frac{V_{inc}}{-i\left\{\kappa_1\left(\frac{\lambda}{L}\right) - \kappa_2\left(\frac{L}{\lambda}\right)\right\} + \frac{4\pi}{9}\left(\frac{L}{\lambda}\right)^2} \tag{2.23}$$

recalling that

$$\kappa_1 \equiv \frac{1}{\pi^2}\left\{2\left[\ln\left(\frac{L}{a}\right)-1\right]-\ln\left(2\left[\ln\left(\frac{L}{a}\right)-1\right]\right)\right\}$$

$$\kappa_2 \equiv \left(\frac{2}{3}\left[\ln\left(\frac{L}{a}\right)\right]\right) \qquad (2.24)$$

By comparison with equation (1.16), repeated below for convenience, we see that the parameters κ are indeed constants of order 1 multiplied by weak (logarithmic) functions of the dimensions of the wire.

$$|I_0| = \frac{|V_0|}{\mu_0 c} \propto \frac{|V_{inc}|}{\mu_0 c\sqrt{\left(\kappa_1\left(\frac{\lambda}{L}\right)-\kappa_2\left(\frac{L}{\lambda}\right)\right)^2+\kappa_3\left(\frac{L}{\lambda}\right)^4}}$$

The resonant wavelength occurs when the imaginary part of the denominator vanishes:

$$\left(\frac{L_{res}}{\lambda}\right)^2 = \frac{\left\{2\left[\ln\left(\frac{L_{res}}{a}\right)-1\right]-\ln\left(2\left[\ln\left(\frac{L_{res}}{a}\right)-1\right]\right)\right\}}{\pi^2\left(\frac{2}{3}\left[\ln\left(\frac{L}{a}\right)\right]\right)} \qquad (2.25)$$

$$\lim_{L/a\to\infty}\left(\frac{L_{res}}{\lambda}\right) = \sqrt{\frac{3}{\pi^2}} \approx 0.55$$

The magnitude of the current is shown in figure 17 as a function of the normalized wavelength, for the same dimensions used previously. We see that there is a strong resonant behavior. The half-power bandwidth is about 0.6 in normalized units, corresponding to a quality factor 'Q' of around 4: the wire scatters about 1/4 of the amount of energy it stores each cycle.

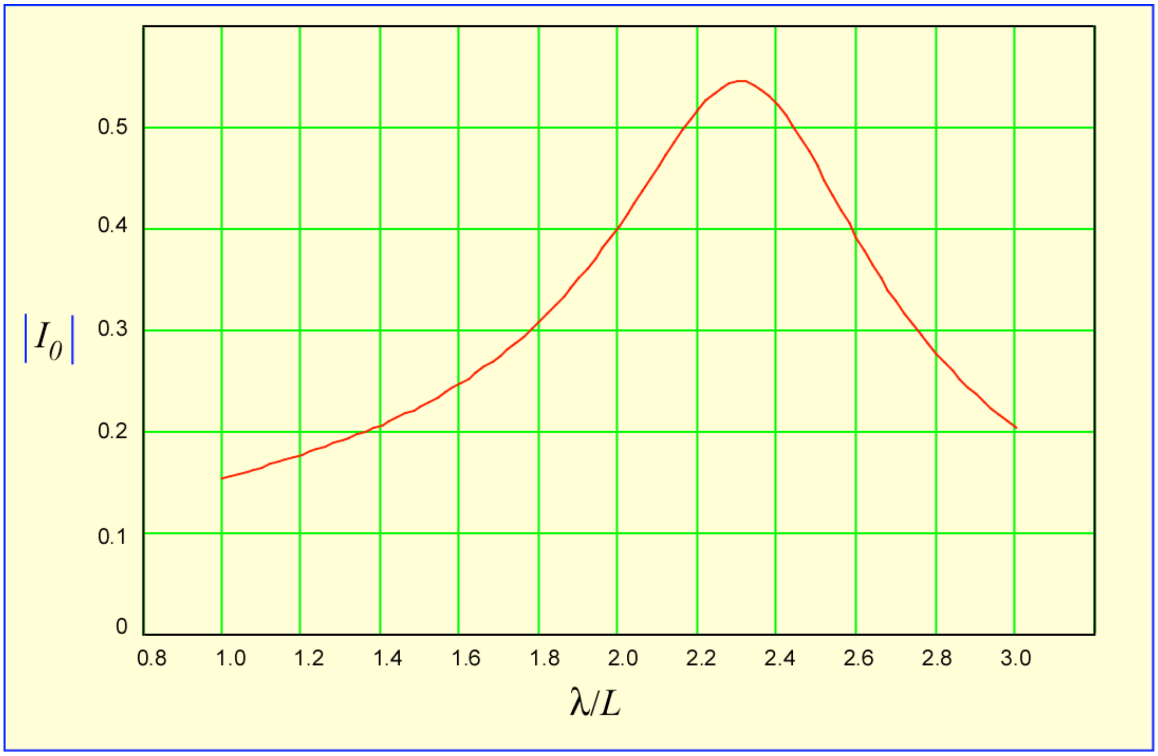

Figure 17: magnitude of induced current vs. normalized wavelength, for L=0.1 m, a=0.001 m (arbitrary units). Resonance at $(\lambda/L)\approx 2.3 \rightarrow (L/\lambda) = 0.44$

Figure 17 shows the same data in the complex plane. The general shape of the current characteristic is the same as that we predicted based on scaling arguments in section I (figure 7). However, we can see that the location of the maximum current is actually slightly displaced from the wavelength at which the current is pure real. This is because the real term in the denominator increases rather rapidly as the wavelength decreases; we get more current by increasing the wavelength a bit even at the cost of a small imaginary part in the denominator.

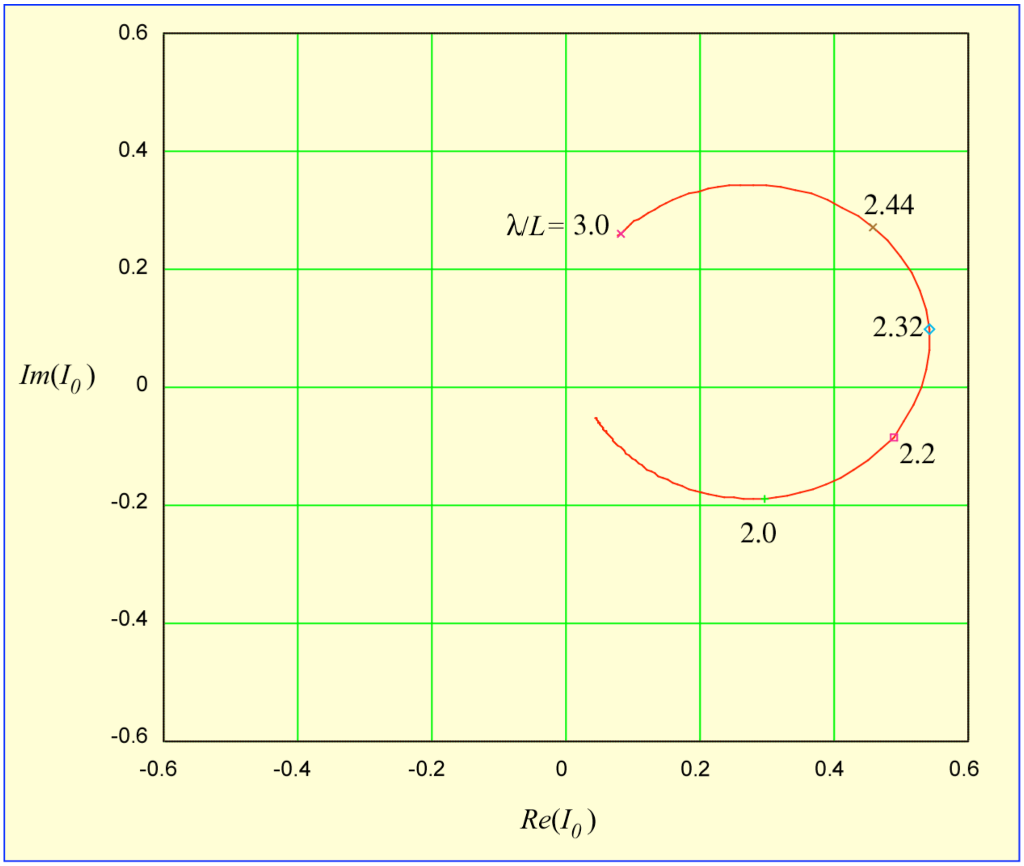

Figure 18: complex current as a function of (normalized) wavelength

2. Wire Dipole Antennas

We can now exploit the results obtained for the current on a wire to analyze the operation of a very common antenna: the wire dipole. To construct such a dipole we cut our test wire in the center and attach each half to a connection wire running perpendicular to the axis of the dipole (figure 19). In the simple case we have examined, where the incident potential is along the axis of the dipole, there will be no coupling of the incident potential to the connection wires, but in the more general case the coupling to an incident potential is very small as long as the connection wires are parallel and closely spaced. Thus we can assume that the scalar potential is constant along the connection wires: the voltage applied to the load is the difference between the scalar potentials at the connection points.

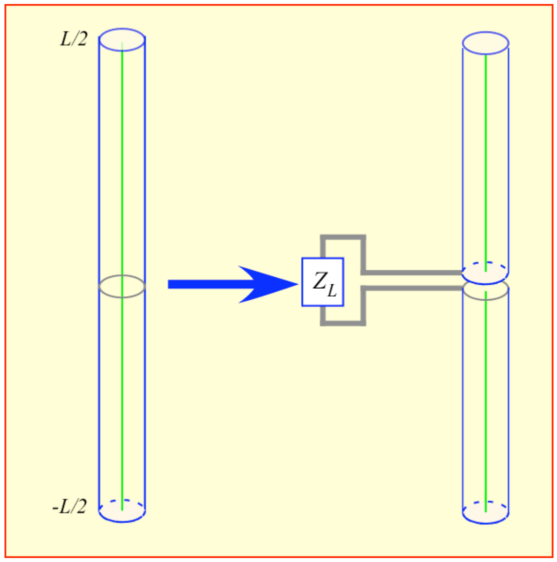

Figure 19: slice a wire in the middle and connect each half to one side of a parallel transmission line to create a dipole antenna

Assuming that everything is linear, we can analyze the behavior of this system by treating the extreme cases where the load is an open circuit or a short circuit. We can use the results of section (1) if we are willing to ignore the hopefully modest effects of the parasitic capacitance and inductance associated with the slice in the wire and connection wires.

When a short circuit load is placed at the antenna terminals, the scalar potential is constant across the gap: that is, the solution is very similar to that we've already obtained for a continuous wire. Therefore we find, assuming again a quadratic current distribution:

$$I_{s.c.} \approx -I_0\left(L, A_{inc}\right) = \frac{-V_0\left(L, A_{inc}\right)}{\mu_0 c} \tag{2.26}$$

where V_0 is taken from equation (2.22). The negative sign arises from the fact that the current is defined as coming out of the antenna instead of entering it (see the equivalent circuit in figure 20).

The open-circuit voltage is a bit more subtle. In this case no current flows into the load, so one can treat the problem as two identical wires of half the length of the dipole, each with a locally quadratic current distribution and linear charge (figure 20). The maximum current in each wire is about that appropriate to a wire of half the length of the dipole; the open-circuit voltage is the difference in the scalar potential between the two adjacent ends.

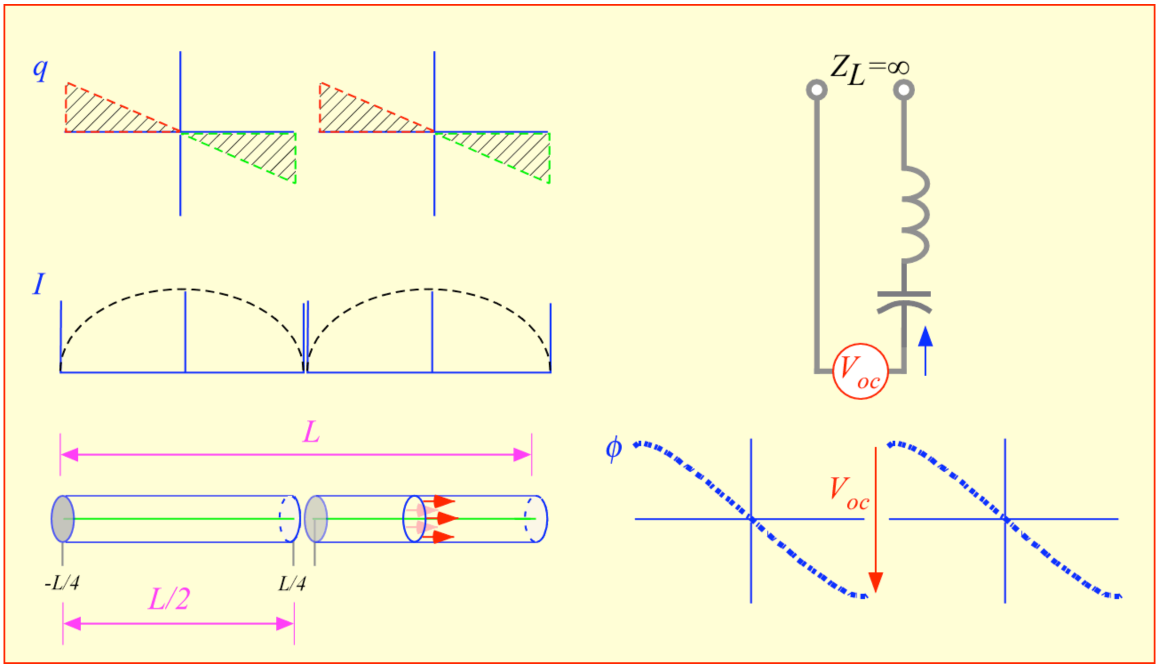

Figure 20: Approximate current and charge distributions for the case of an open-circuited dipole

The scalar potential is that due to two adjacent quadratic current distributions. A quantitative appreciation of the situation would integrate over both distributions, accounting for the separation between the segments of the antenna, and thus incorporating a certain amount of parasitic capacitance (and mutual inductance) between the two segments. However, this is a bit more complex than needed for a first estimate of the behavior of the antenna; in particular, if we match the electric field in the center of a segment to the incident field, we can have confidence that the effect of the distant segment is rather modest. So let us use exactly the expressions we have used previously, with the half-length *L/2* substituted for *L*:

$$V_{segment,\phi} = iV_{0,seg}\left(\frac{2\lambda}{L}\right)\kappa_1\left[\frac{L}{2}\right];$$
$$V_{segment,in} = -iV_{0,seg}\left(\frac{L}{2\lambda}\right)\kappa_2\left[\frac{L}{2}\right] \qquad (2.27)$$

The delayed contribution to the potential is a bit more subtle: the integral (2.16) is independent of the distance, so in this level of approximation all currents contribute equally to the delayed potential. The scalar potential behaves in a similar fashion: the delayed contribution to the electric field for the two segments is similar to that for a single segment with the same peak current. Therefore, the electric field has the same relation to the peak current as for the shorted dipole, and the full length L must be used:

$$E_{segment,d} = -\frac{4\pi V_{0,seg}}{9}\left(\frac{L}{\lambda^2}\right) \tag{2.28}$$

The voltage across each individual segment is therefore the product of this field with the segment length (*L/2*):

$$V_{sc,d} = -\frac{4\pi V_{0,seg}}{9}\left(\frac{L}{\lambda^2}\right)\left(\frac{L}{2}\right) = -\frac{2\pi V_{0,seg}}{9}\left(\frac{L}{\lambda}\right)^2 \tag{2.29}$$

We now proceed as before to match the scattered voltage on the segment to the incident voltage on the segment, which is half of the voltage on the antenna as a whole:

$$V_{sc} = \frac{-V_{inc}}{2} = iV_{0,seg}\left\{\left(\frac{2\lambda}{L}\right)\kappa_1\left[\frac{L}{2}\right] - \left(\frac{L}{2\lambda}\right)\kappa_2\left[\frac{L}{2}\right]\right\} - \frac{2\pi V_0}{9}\left(\frac{L}{\lambda}\right)^2 \tag{2.30}$$

We solve for the characteristic voltage of the segment:

$$V_{0,seg} = \frac{-V_{inc}}{\left\{i\left\{\left(\frac{4\lambda}{L}\right)\kappa_1\left[\frac{L}{2}\right] - \left(\frac{L}{\lambda}\right)\kappa_2\left[\frac{L}{2}\right]\right\} - \frac{4\pi}{9}\left(\frac{L}{\lambda}\right)^2\right\}} \tag{2.31}$$

Inserting this value into equation (2.27), and referencing the incident voltage to the length of the dipole L, we obtain the open-circuit voltage as a function of the incident potential and antenna length:

$$V_{oc} \approx \frac{-V_{inc}\kappa_1\left[\frac{L}{2}\right]}{\left\{2\kappa_1\left[\frac{L}{2}\right] - \frac{1}{2}\left(\frac{L}{\lambda}\right)^2\kappa_2\left[\frac{L}{2}\right]\right\} + \frac{2\pi i}{9}\left(\frac{L}{\lambda}\right)^3} \tag{2.32}$$

Note that when L/λ is small compared to 1, the first term in the denominator dominates the expression and we find very simply that:

$$V_{sc,\phi}(segment) \approx \frac{-V_{inc}\kappa_1\left[\frac{L}{2}\right]}{2\kappa_1\left[\frac{L}{2}\right]} = \frac{-V_{inc}}{2} \qquad (2.33)$$

The open-circuit voltage for a short antenna is just half of the incident voltage ($E_{inc}L$). As the antenna grows longer the open-circuit voltage increases. The imaginary part of the short-circuit current and the real part of the open-circuit voltage as a function of frequency are depicted in figure 21, using the same dimensions we have employed before: $L = 0.1$ m, $a = 0.001$ m, both for a fixed incident voltage of 1 volt. The open-circuit voltage is about 1/2 of the incident voltage at frequencies well below resonance, growing gradually as frequency increases. We see that the current is roughly linear in frequency at low frequencies, growing more rapidly as resonance approaches. (Note that the maximum value of current occurs just a bit below the frequency at which the current becomes pure real, as noted in connection with figure 18 above.)

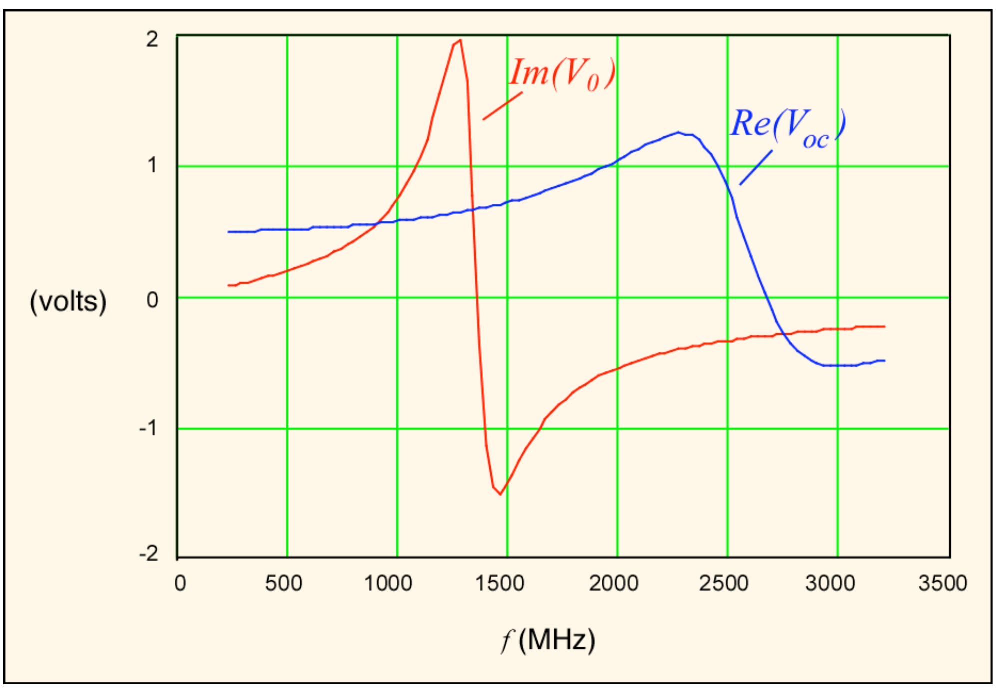

Figure 21: open-circuit voltage V_{oc}, and V_0 (Z_0•short-circuit current); L=0.1 m and a=0.001 m (L/a=100); V_{inc}=1

Now that we have obtained both the short-circuit current and the open-circuit voltage, we can construct an equivalent circuit for the antenna using Thevenin's theorem (figure 22): the equivalent source impedance is the ratio of the open-circuit voltage (2.33) and the short-circuit current (2.26).

$$Z_{eq} = \frac{V_{oc}}{I_{sc}} = \mu_0 c \frac{\kappa_1\left[\frac{L}{2}\right]\left\{-i\left\{\left(\frac{\lambda}{L}\right)\kappa_1[L] - \left(\frac{L}{\lambda}\right)\kappa_2[L]\right\} + \frac{4\pi}{9}\left(\frac{L}{\lambda}\right)^2\right\}}{\left\{2\kappa_1\left[\frac{L}{2}\right] - \frac{1}{2}\left(\frac{L}{\lambda}\right)^2\kappa_2\left[\frac{L}{2}\right]\right\} + \frac{2\pi i}{9}\left(\frac{L}{\lambda}\right)^3} \tag{2.34}$$

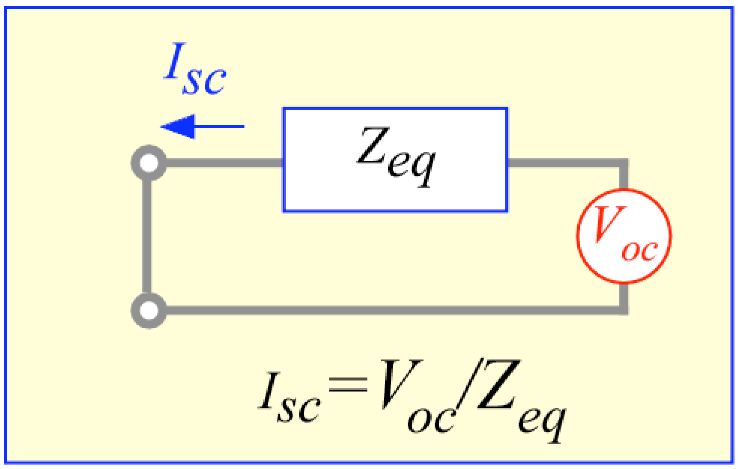

Figure 22: derivation of equivalent source impedance

For small (L/λ) we can approximate:

$$Z_{eq} \approx \frac{\mu_0 c}{2} \frac{\left\{-i\left\{\left(\frac{\lambda}{L}\right)\kappa_1[L]\right\} + \frac{4\pi}{9}\left(\frac{L}{\lambda}\right)^2\right\}}{1 + \frac{\frac{\pi i}{9}\left(\frac{L}{\lambda}\right)^3}{\kappa_1\left[\frac{L}{2}\right]}}$$

$$= \frac{\mu_0 c}{2}\left\{-i\left\{\left(\frac{\lambda}{L}\right)\kappa_1[L]\right\} + \frac{4\pi}{9}\left(\frac{L}{\lambda}\right)^2\right\}\left\{1 - \frac{\frac{\pi i}{9}\left(\frac{L}{\lambda}\right)^3}{\kappa_1\left[\frac{L}{2}\right]}\right\} \tag{2.35}$$

$$\approx \frac{\mu_0 c}{2}\left[-i\left\{\left(\frac{\lambda}{L}\right)\kappa_1[L]\right\} + \frac{\pi}{9}\left(\frac{L}{\lambda}\right)^2\left[4 - \frac{\kappa_1[L]}{\kappa_1\left[\frac{L}{2}\right]}\right]\right]$$

Now recall that the factor κ_1 is essentially of the form $ln(L/a)$; for large L/a κ_1 changes very slowly as a function of L, so the ratio $\kappa_1(L)/\kappa_1(L/2)$ is nearly equal to 1. Thus for practical values we can further simplify equation (2.35):

$$Z_{eq} \approx \underbrace{\mu_0 c \frac{\pi}{6}\left(\frac{L}{\lambda}\right)^2}_{R_{rad}} - \underbrace{i\mu_0 c\left\{\frac{1}{2}\left(\frac{\lambda}{L}\right)\kappa_1[L]\right\}}_{\frac{1}{i\omega C_{eq}}} \tag{2.36}$$

The real part of the impedance – the ***radiation resistance*** – is proportional to the square of the normalized antenna length, and would be about 50 Ω at resonance ($L/\lambda \approx 1/2$) in this approximation. However, as the wavelength decreases, one can no longer ignore the contribution of the vector potential as was done in deriving (2.36). The real and imaginary parts of the impedance from the full expression (2.34) are shown in figure 23 as a function of frequency. The antenna looks like a capacitance at low frequencies, and passes through a resonance just about 1300 MHz, thereafter becoming inductive and roughly linear in frequency. The maximum in current occurs at about 1.34 GHz ($L/\lambda \approx$ 0.45), where the antenna is slightly capacitive: $Z(\Omega) \approx 69 - j19$. The impedance is pure real at about 1.365 GHz ($L/\lambda \approx 0.455$), where the radiation resistance is 74 ohms. When the antenna approaches one wavelength in length, the segments undergo resonance, creating a maximum in the real impedance.

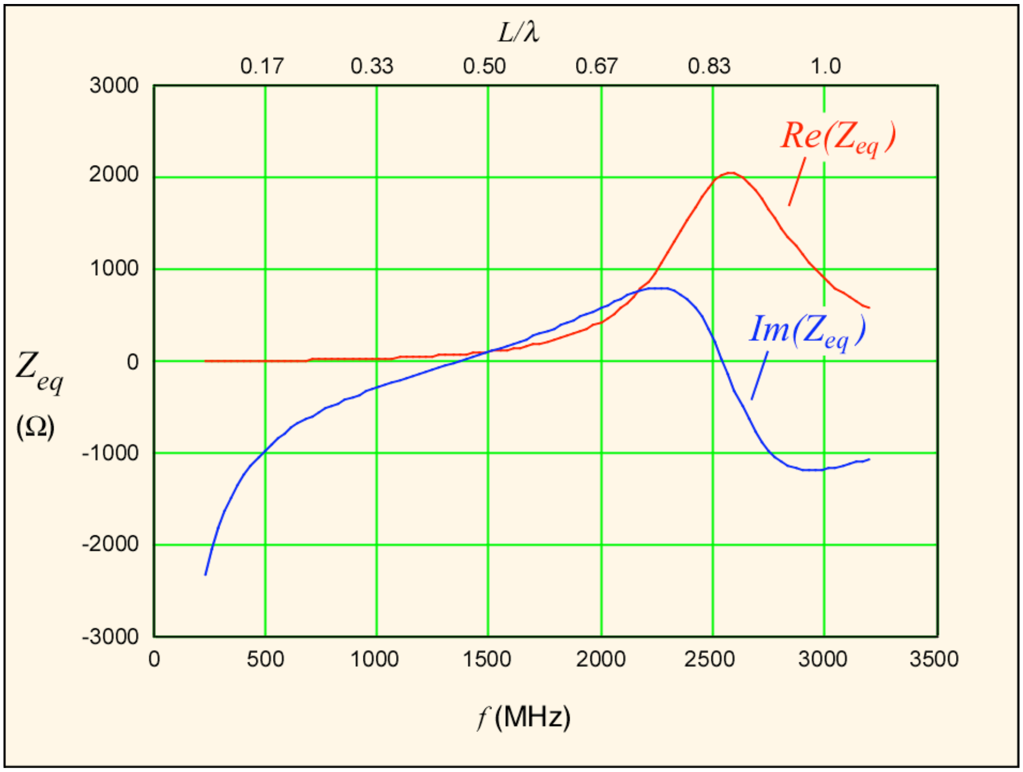

Figure 23(a): equivalent source impedance vs. frequency, conditions as figure 21

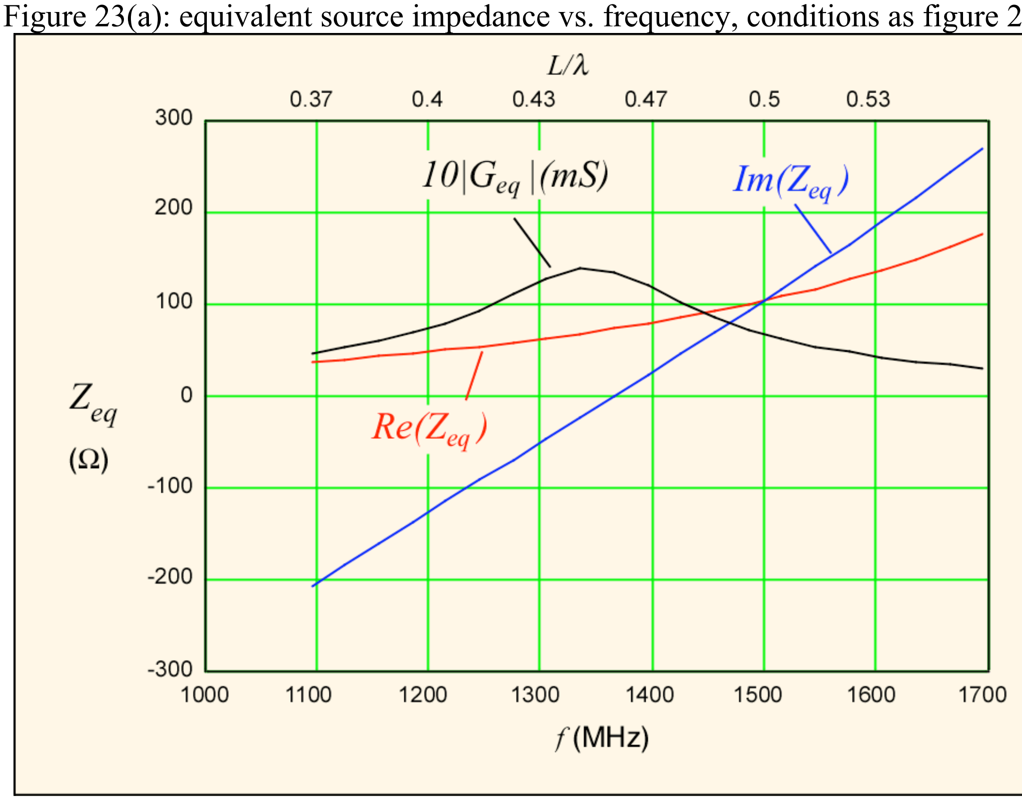

Figure 23(b): closeup of the data near resonance; $|G|$ is the magnitude of the conductance

How do these analytic estimates compare to numerical calculations of the input impedance of a wire antenna? In figure 24, we show the input impedance of a wire antenna with a fixed radius of 0.0005λ as a function of the normalized antenna length, computed using equation (2.34) above, compared to numerical results for the same geometry from Stutzman and Thiele [1]. It is apparent that agreement is excellent for antenna lengths less than half a wavelength, and remarkably good even for antennas up to 1 wavelength long, where the approximations employed in deriving equation (2.34) are questionable.

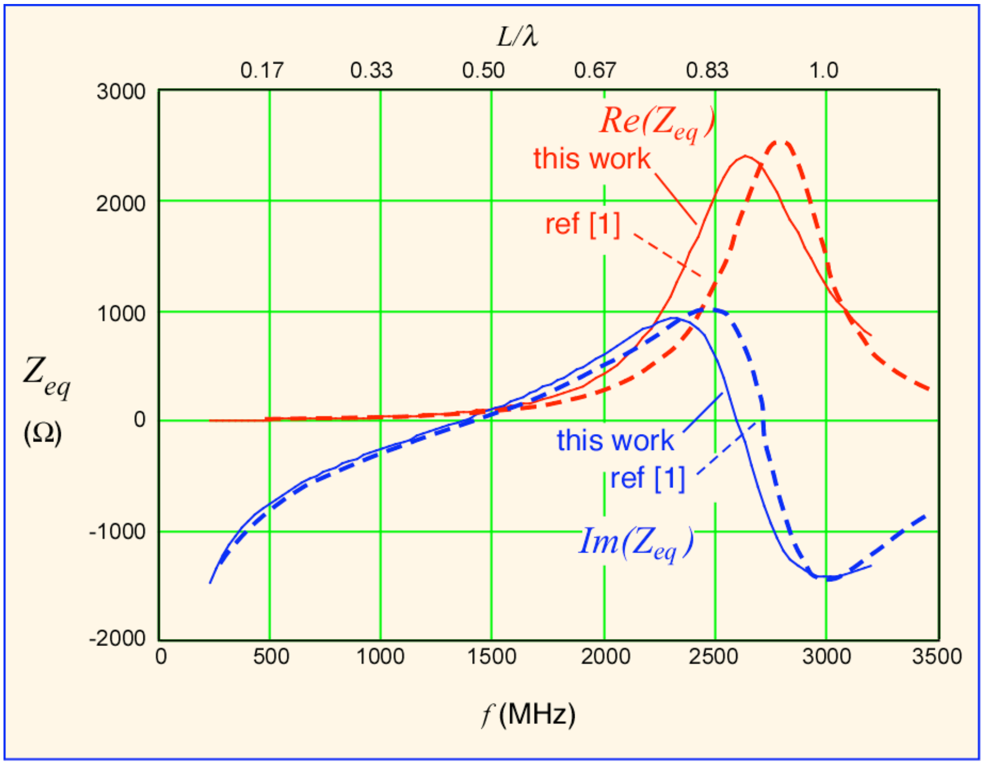

Figure 24: input impedance of dipole antenna vs. frequency, for $a = 0.0005\lambda$; solid lines are results of equation (2.34), dashed lines taken from [1] (1st edition) figures 5-5,-6.

Expressing the impedance in terms of the normalized length is useful for dealing with a physical antenna, but when thinking in electrical terms it may be preferable to make the frequency dependence explicit. In equation (2.37) we have rewritten the imaginary part of the impedance (ignoring the radiation resistance altogether) in terms of a part roughly proportional to the frequency and another inversely proportional to frequency, which can be approximately regarded as an equivalent capacitance and inductance.

$$Z_{eq} = i\omega \left[\underbrace{\mu_0 c \frac{\left(\frac{L}{2\pi c}\right)\kappa_1\left[\frac{L}{2}\right]\kappa_2\left[L\right]}{\left\{2\kappa_1\left[\frac{L}{2}\right]-\frac{1}{2}\left(\frac{L}{\lambda}\right)^2\kappa_2\left[\frac{L}{2}\right]\right\}}}_{\ell_{eff}} \right] - i\frac{1}{\omega}\left[\underbrace{\frac{\left\{2\kappa_1\left[\frac{L}{2}\right]-\frac{1}{2}\left(\frac{L}{\lambda}\right)^2\kappa_2\left[\frac{L}{2}\right]\right\}}{\mu_0 c^2\kappa_1\left[\frac{L}{2}\right]\left(\frac{2\pi}{L}\right)\kappa_1\left[L\right]}}_{C_{eff}} \right]^{-1} \quad (2.37)$$

Note that these equivalent parameters are not truly constant in frequency. In the limit of low frequency and very large (L/a) we obtain:

$$C_{eff} = \frac{\pi L}{2\mu_0 c^2 \left[\ln\left(\frac{L}{a}\right)\right]} \quad (2.38)$$

$$\ell_{eff} = \left(\frac{\mu_0 L}{6\pi}\right)\left[\ln\left(\frac{L}{a}\right)\right] \quad (2.39)$$

The capacitance and inductance are both proportional to the antenna length, but have opposite dependences on the ratio of length to radius, explaining why the resonant frequency is only weakly dependent on (L/a).

The equivalent inductance and capacitance vs. frequency for our sample antenna are shown in figure 25. It is apparent that using an equivalent circuit with fixed capacitance and inductance is not unreasonable, but will introduce errors of around 20-30% in the resulting impedance (depending on what fixed values are chosen).

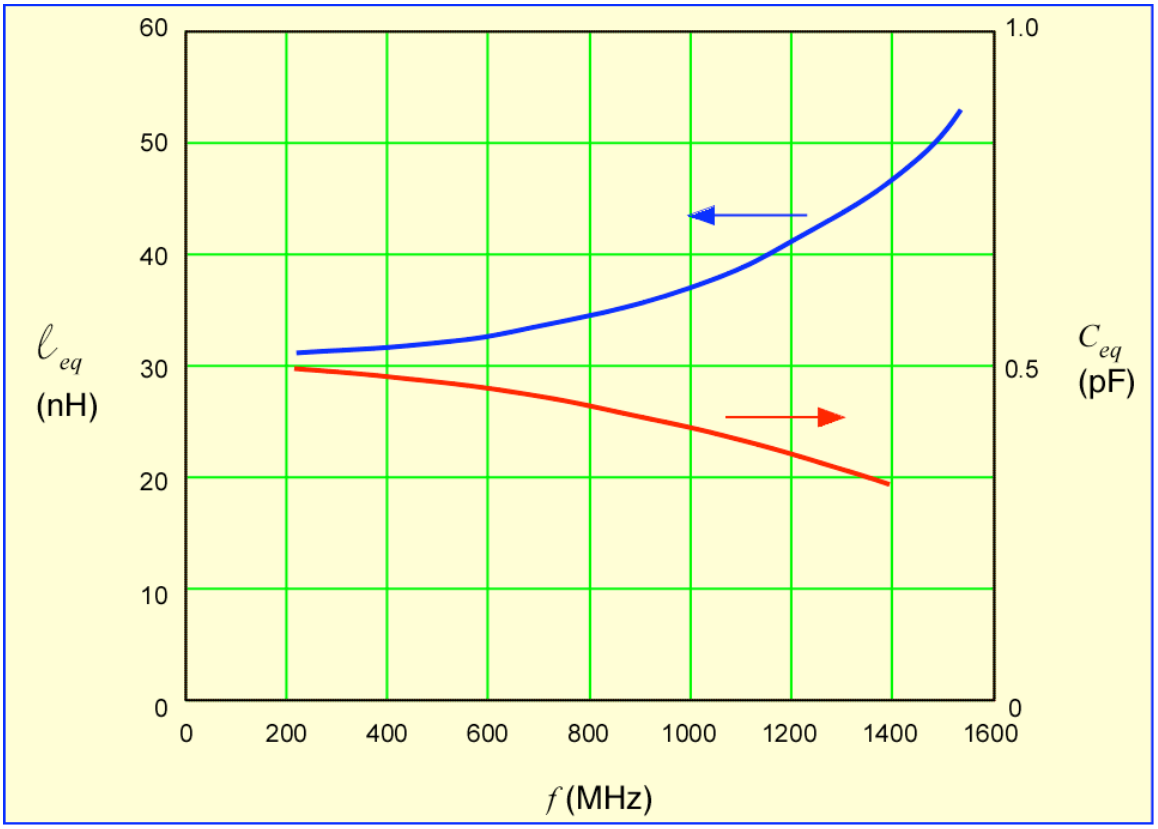

Figure 25: equivalent capacitance and inductance vs. frequency for L=0.1 m, L/a=100

To summarize this section, we've found:

- A dipole antenna can be treated using the results of section (1) for a wire combined with Thevenin's theorem.
- The impedance of the antenna looks roughly like a series combination of an inductor and a capacitor, but the values of the components vary somewhat with frequency. Nevertheless, this approach provides a simple means of understanding the resonant behavior of the antenna.
- The real (radiation) resistance arises mainly from the quadrature vector potential resulting from the average current over the antenna: currents that don't cancel lead to radiation! The radiation resistance is reduced by a smaller contribution from the scalar potential; though we have not discussed this fact in detail, the cancellation of the scalar and vector potentials in the far field ensures that no power is radiated along the axis of the antenna.
- The radiation resistance is not constant, but varies about quadratically for very short antennas, and rises to a peak around an antenna length of about 1 wavelength. This peak is the result of the large currents flowing in the individual segments at their resonant frequency, which give rise to charge accumulation at the segment ends and a high open-circuit voltage.

3. Afterword:

In part I, we made heuristic arguments about how potentials, currents, and charges ought to interact in a wire antenna subject to an impinging field. In part II, we demonstrated that the expressions one arrives at by actually integrating the charge and current distributions to obtain the potentials do indeed scale in the promised fashion. The integrations lead to a set of complex-looking constant factors, but a careful inspection shows that all these constants are all close relatives of $ln(L/a)$: that is, the fundamental physics is the logarithmic effect of charges and currents in a cylindrical region have on their neighbors.

We saw that the impedance of an antenna is the direct result of the interactions of local charges and currents. It is not necessary to consider the power flow in the far field, or to explicitly calculate the power balance at the surface of the antenna, though of course such methods are perfectly valid approaches to the problem. The contribution of the vector potential to the radiation resistance is of a particularly simple form, and is accurate enough by itself to serve for a first estimate. However, accurate calculations require that a third-order term in the scalar potential integral be included. In the far field, this scalar potential term acts to cancel the vector potential along the axis of the antenna, giving rise to the observation that only the transverse part of the vector potential conveys power (more commonly though equivalently phrased in terms of the electric field) [5]. This result is usually obtained by imposing the Lorentz gauge on the fields, which has the advantage of mathematical elegance but the disadvantage of concealing the separate actions of charges and currents.

The increase in real impedance at antenna lengths approaching one wavelength can be seen as the result of the high open-circuit voltage as the individual segments of the antenna approach resonance. As we near resonance, the near cancellation of the scalar and vector components of the electric field means that large currents flow in the segments, and copious quantities of charge accumulate at the segment ends, leading to an enhanced scalar potential and thus a large open-circuit voltage.

The resulting predictions (figure 24) are remarkably accurate, given the crudity with which the complex exponential e^{-ikr} has been treated. For example, in an antenna a half wavelength long, the charges at one end should be multiplied by $e^{-i\pi} = -1$ to obtain their effect on the charges at the opposite end, whereas we have used instead $(1-i\pi)$, rather seriously in error! We have also used a primitive estimate of the charge distribution. The very serviceable results obtained show us that the most important influences on the potentials result from nearby charges, so that erroneous treatment of more distant effects has only a modest effect on the result. The biggest error in our treatment, using a charge distribution with inadequate charge at the ends of the wire, was finessed by extrapolating to the wire ends (equations (2.8) and (2.9)), effectively correcting the error. It can be shown that the actual amount of charge involved, and the corresponding change in current flow, is very tiny for reasonable geometries, and mainly affects only the potential near the ends, explaining why our fudge doesn't invalidate the corresponding vector potential calculation.

Finally, we have obtained all these results with recourse whatsoever to the magnetic field **B**, or its relative the Poynting vector **E x B**. In fact, no curls or cross-products have been needed at all. The magnetic field is a mathematical construct that arises naturally in the consideration of loops of wire, because of the equivalence of the line integral over a closed curve and the integral of the relevant curl over the enclosed surface. It is neither necessary nor (in the current author's view) desirable in analyzing linear structures.

As an entertaining aside, it is worth a moment to consider the magnitudes of these charges and currents upon which we lavish so much devotion. Around resonance the current in a wire is on the order of $3V_{inc}/\mu_0 c$. For an incident field of 10 V/m at 1.3 GHz on our 10-cm antenna, that's 3/377 ≈ 8 mA. Let us assume that the current is concentrated in a surface region about 1 micron deep around the 3.14-mm periphery of the wire. The current density in the surface is thus about 250 A/cm^2. Assuming the electron density in the wire is about 6×10^{22}/cm^3 (appropriate to aluminum), the peak electron velocity is around $250/([1.6 \times 10^{-19}])(6 \times 10^{22}) = 0.03$ cm/s. In one half of an RF cycle, the electrons move $(0.03)(4 \times 10^{-10}) = 10^{-11}$ cm. Everything we depend on in a receiving antenna results from the motion of electrons over distances ***comparable to the size of an atomic nucleus***! The corresponding charge accumulations are similarly miniscule: the aforesaid current of 8 mA transports a total charge of 6×10^{-12} coulombs, or about 4×10^{7} electron charges from one half of the wire to the other. Since our 10 cm x 1 mm wire has around 5×10^{21} electrons in it, the charge transport is equivalent to moving about 1 of every hundred million million electrons (10^{14}) along the wire. Charge neutrality is indeed a very powerful principle, and even the smallest violation is readily revealed and amenable to our use.

Acknowledgments:
The author would like to thank Prof. William Schaffer, Prof. George Vendelin, and Prof. Shyn-kang Jeng, who reviewed earlier versions of this work.

References

1] **Antenna Theory and Design (2nd Ed)**, W. Stutzman & G. Thiele, Wiley 1997
2] **Antenna Theory, Analysis and Design (2nd Ed)**, C. Balanis, Wiley 1996
3] **Antennas (3rd Ed)**, J. Kraus & R. Marhefka, McGraw-Hill 2001
4] **Collective Electrodynamics**, C. Mead, MIT Press 2002
5] **RF Engineering for Wireless Networks**, D. Dobkin, Elsevier 2004, esp. Appendix 4

Appendix:

Here we present the guts of several derivations mercifully absent from the main text.

1] The point of zero gradient of the scalar potential (equations (2.8) and (2.9):

We first assert that the location of this interesting point is reasonably near the ends of the wire:

$$\eta_{\max} = \left(\frac{L}{2}\right)(1-x); \quad x << 1; K = \frac{V_0 \lambda}{i\pi^2 L^2}$$

We then evaluate the potential near the end:

$$\phi(\eta) \approx K\left(\frac{L}{2}\right)(1-x)\left\{\ln\left(\frac{(L/2)^2 - \left(\left(\frac{L}{2}\right)(1-x)\right)^2}{a^2}\right) + \ln(2) - 2\right\}$$

$$= K\left(\frac{L}{2}\right)(1-x)\left\{\ln\left(\frac{\left(\frac{L}{2}\right)^2\left(1-(1-x)^2\right)}{a^2}\right) + \ln(2) - 2\right\}$$

Because x is small, the logarithm can be simplified, in the process consuming the *ln*(2)*:*

$$\approx K\left(\frac{L}{2}\right)(1-x)\left\{\ln\left(\frac{\left(\frac{L}{2}\right)^2(2x)}{a^2}\right) + \ln(2) - 2\right\} = K\left(\frac{L}{2}\right)(1-x)\left\{\ln\left(\frac{L}{a}\right)^2 + \ln(x) - 2\right\}$$

We then differentiate and set the derivative to 0 to find the maximum or minimum:

$$\frac{d}{dx}(1-x)\left\{\ln\left(\frac{L}{a}\right)^2 + \ln(x) - 2\right\} = \frac{(1-x)}{x} - \ln\left(\frac{L}{a}\right)^2 - \ln(x) + 2 = 0 \approx \frac{1}{x} + \ln\left(\frac{1}{x}\right) = \ln\left(\frac{L}{a}\right)^2 - 2$$

$$\rightarrow x \approx \frac{1}{\ln\left(\frac{L}{a}\right)^2 - 2}$$

2] <u>Evaluating the vector potential (equation 2.11)</u>

This is a laborious but straightforward application of the integrals (2.1) and (2.2):

$$A_{sc}(\eta)=\frac{\mu_0}{4\pi}\int_{-L/2}^{L/2}\frac{I(z)\,dz}{\sqrt{a^2+(z-\eta)^2}}=\frac{\mu_0 I_0}{4\pi}\int_{-L/2}^{L/2}\frac{\left(1-\left(\frac{2z}{L}\right)^2\right)dz}{\sqrt{a^2+(z-\eta)^2}}=\frac{\mu_0 I_0}{4\pi}\int_{-L/2-\eta}^{L/2-\eta}\frac{\left(1-\left(\frac{2u+2\eta}{L}\right)^2\right)du}{\sqrt{a^2+u^2}}$$

where $u=z-\eta$

$$=\frac{\mu_0 I_0}{4\pi}\int_{-L/2-\eta}^{L/2-\eta}\frac{\left(\left(1-\frac{4}{L^2}\eta^2\right)-\frac{4}{L^2}u^2-\frac{8}{L^2}\eta u\right)du}{\sqrt{a^2+u^2}}$$

$$=\frac{\mu_0 I_0}{4\pi}\left\{\left(1-\frac{4}{L^2}\eta^2\right)\int_{-L/2-\eta}^{L/2-\eta}\frac{du}{\sqrt{a^2+u^2}}-\frac{8}{L^2}\eta\int_{-L/2-\eta}^{L/2-\eta}\frac{u\,du}{\sqrt{a^2+u^2}}-\frac{4}{L^2}\int_{-L/2-\eta}^{L/2-\eta}\frac{u^2\,du}{\sqrt{a^2+u^2}}\right\}$$

$$=\frac{\mu_0 I_0}{4\pi}\left\{\begin{array}{l}\left(1-\frac{4}{L^2}\eta^2\right)\ln\left(u+\sqrt{a^2+u^2}\right)\Big|_{-L/2-\eta}^{L/2-\eta}-\frac{8}{L^2}\eta\sqrt{a^2+u^2}\Big|_{-L/2-\eta}^{L/2-\eta}\\ -\frac{2}{L^2}\left[u\sqrt{a^2+u^2}-a^2\ln\left(u+\sqrt{a^2+u^2}\right)\right]_{-L/2-\eta}^{L/2-\eta}\end{array}\right\}$$

$$=\frac{\mu_0 I_0}{4\pi}\left\{\begin{array}{l}\left(1-\frac{4}{L^2}\eta^2\right)\ln\left(\frac{L/2-\eta+\sqrt{a^2+(L/2-\eta)^2}}{-L/2-\eta+\sqrt{a^2+(L/2+\eta)^2}}\right)\\ -\frac{8}{L^2}\eta\left[\sqrt{a^2+(L/2-\eta)^2}-\sqrt{a^2+(L/2+\eta)^2}\right]\\ -\frac{2}{L^2}\left[\begin{array}{l}(L/2-\eta)\sqrt{a^2+(L/2-\eta)^2}+(L/2+\eta)\sqrt{a^2+(L/2+\eta)^2}\\ -a^2\ln\left(\frac{L/2-\eta+\sqrt{a^2+(L/2-\eta)^2}}{-L/2-\eta+\sqrt{a^2+(L/2+\eta)^2}}\right)\end{array}\right]\end{array}\right\}$$

$$=\frac{\mu_0 I_0}{4\pi}\left\{\begin{array}{l}\left(1-\frac{4}{L^2}\eta^2-\frac{2}{L^2}a^2\right)\ln\left(\frac{L/2-\eta+\sqrt{a^2+(L/2-\eta)^2}}{-L/2-\eta+\sqrt{a^2+(L/2+\eta)^2}}\right)\\ +\left[\left(\frac{6}{L^2}\eta-\frac{1}{L}\right)\sqrt{a^2+(L/2+\eta)^2}-\left(\frac{6}{L^2}\eta+\frac{1}{L}\right)\sqrt{a^2+(L/2-\eta)^2}\right]\end{array}\right\}$$

3] <u>Integration of the fit to the scattered vector potential (equation (2.15)):</u>

First we realize that the first bracket and last term are just constants with respect to the variable of integration, and name them *a* and *b*:

$$\mathrm{A}_{\mathrm{sc}} \approx \frac{\mu_0 I_0}{4\pi}\left\{\left[\ln\left(\left(\frac{L}{a}\right)^2\right)-3\right]\left[1-\left(\frac{2\eta}{L}\right)^2\right]+2\right\}$$

$$= \frac{\mu_0 I_0}{4\pi}\left\{a\left[1-\left(\frac{2\eta}{L}\right)^2\right]+b\right\}$$

Performing the integration we obtain:

$$\int_{-L/2}^{L/2} \frac{\mu_0 I_0}{4\pi}\left\{a\left[1-\left(\frac{2\eta}{L}\right)^2\right]+b\right\}d\eta = \frac{\mu_0 I_0 L}{4\pi}\left(b+\frac{2}{3}a\right)$$

$$\omega\int \mathrm{A}_{\mathrm{sc}} = \frac{2\pi c}{\lambda}\frac{\mu_0 I_0 L}{4\pi}\left(2+\frac{2}{3}\left[\ln\left(\left(\frac{L}{a}\right)^2\right)-3\right]\right) = V_0\frac{L}{\lambda}\left(1+\frac{1}{3}\left[\ln\left(\left(\frac{L}{a}\right)^2\right)-3\right]\right)$$

Multiplying by the angular frequency to get the electric field contribution, we find:

$$\omega\int \mathrm{A}_{\mathrm{sc}} = \frac{2\pi c}{\lambda}\frac{\mu_0 I_0 L}{4\pi}\left(2+\frac{2}{3}\left[\ln\left(\left(\frac{L}{a}\right)^2\right)-3\right]\right) = V_0\frac{L}{\lambda}\left(1+\frac{1}{3}\left[\ln\left(\left(\frac{L}{a}\right)^2\right)-3\right]\right)$$

$$= V_0\frac{L}{\lambda}\left(1+\left[\frac{2}{3}\ln\left(\frac{L}{a}\right)-1\right]\right) = V_0\frac{L}{\lambda}\left(\frac{2}{3}\ln\left(\frac{L}{a}\right)\right)$$